

ISOPERIMETRY IN SUPERCRITICAL BOND PERCOLATION IN DIMENSIONS THREE AND HIGHER

JULIAN GOLD

ABSTRACT. We study the isoperimetric subgraphs of the infinite cluster \mathbf{C}_∞ for supercritical bond percolation on \mathbb{Z}^d with $d \geq 3$. Specifically, we consider the subgraphs of $\mathbf{C}_\infty \cap [-n, n]^d$ which have minimal open edge boundary to volume ratio. We prove a shape theorem for these subgraphs, obtaining that when suitably rescaled, these subgraphs converge almost surely to a translate of a deterministic shape. This deterministic shape is itself an isoperimetric set for a norm we construct. As a corollary, we obtain sharp asymptotics on a natural modification of the Cheeger constant for $\mathbf{C}_\infty \cap [-n, n]^d$. This settles a conjecture of Benjamini for the version of the Cheeger constant defined here.

CONTENTS

1. Introduction and Results	3
1.1. Motivation	3
1.2. Results	4
1.3. History and discussion	5
1.4. Outline	6
1.5. Open problems	7
1.6. Acknowledgements	7
2. Definitions and notation	8
2.1. Graphs	8
2.2. Percolation	9
2.3. A preliminary setting of convergence	9
2.4. Some geometric measure theory	10
3. The norm $\beta_{p,d}$ and the Wulff crystal	10
3.1. Discrete cylinders, cutsets and connectivity	11
3.2. Defining the norm	12
3.3. The chosen rotation and properties of $\beta_{p,d}$	15
4. Concentration estimates for $\beta_{p,d}$	18
5. Consequences of concentration estimates	21
5.1. Lower bounds for cuts in thin cylinders	21
5.2. Upper bounds on $\widehat{\Phi}_n$, or efficient carvings of ice	23
6. Coarse graining	28
6.1. The construction of Zhang	29
6.2. Webbing	34
6.3. A Peierls argument	38
7. Contiguity	41
7.1. A contiguity argument	43
7.2. Closeness to sets of finite perimeter	46
8. Proof of main results	47
8.1. Setup, the reduced boundary and a covering lemma	47
8.2. Local modification of each $\partial^\omega G_n$	48
8.3. Proof of main results	54
Appendix A. Tools from percolation, graph theory and geometry	58
A.1. Tools from percolation	58
A.2. Using tools from percolation	59
A.3. Tools from graph theory	61
A.4. Approximation and miscellany	61
References	63

1. INTRODUCTION AND RESULTS

1.1. Motivation. Isoperimetric problems, namely the problem of finding a set of given size and minimal boundary measure, have been studied for millennia [9]. In the continuum, such problems are the subject of geometric measure theory and the calculus of variations. Isoperimetric inequalities give a lower bound on the boundary measure of a set in terms of the volume measure of the set. Their applications in mathematics range from concentration of measure to PDE theory.

Isoperimetric problems are also well-studied in the discrete setting. One can encode isoperimetric inequalities for graphs in the *Cheeger constant*, or modifications thereof. For a graph G , define the Cheeger constant of G to be

$$(1.1) \quad \Phi_G := \min \left\{ \frac{|\partial_G H|}{|H|} : H \subset G, 0 < |H| \leq |G|/2 \right\}$$

where $\partial_G H$ is the edge boundary of H in the graph G and where $|H|$ and $|G|$ respectively denote cardinalities of the vertex sets of H and G . This constant was originally introduced for manifolds in Cheeger's thesis [24], in which the Cheeger constant was used to give a lower bound on the smallest positive eigenvalue of the negative Laplacian. Its discrete analogue, introduced by Alon in [3], plays a similar role in spectral graph theory (see for instance Chapter 2 of [25]). Indeed, Cheeger's inequality and its variants are used to study mixing times of random walks and more general Markov chains. Ultimately, the Cheeger constant provides one of many ways to study the geometry of a graph.

Broadly, the goal of this paper is to explore the geometry of random graphs arising from bond percolation on \mathbb{Z}^d . Specifically, we view \mathbb{Z}^d as a graph, with edge set $E(\mathbb{Z}^d)$ determined by nearest-neighbor pairs, and we form the probability space $(\{0, 1\}^{E(\mathbb{Z}^d)}, \mathcal{F}, \mathbb{P}_p)$, where \mathcal{F} denotes the product σ -algebra on $\{0, 1\}^{E(\mathbb{Z}^d)}$ and where \mathbb{P}_p is the product Bernoulli measure associated to the *percolation parameter* $p \in [0, 1]$. Elements $\omega = (\omega_e)_{e \in E(\mathbb{Z}^d)}$ of our probability space are referred to as *percolation configurations*. We say that an edge $e \in E(\mathbb{Z}^d)$ is *open* in the configuration ω if $\omega_e = 1$; we say that an edge is *closed* otherwise. The collection of open edges determine a random subgraph of \mathbb{Z}^d ; the connected components of this subgraph are referred to as *open clusters*. It is well-known (see Grimmett [37] for details) that for $d \geq 2$, bond percolation exhibits a phase transition. That is, there exists a $p_c(d) \in (0, 1)$ such that whenever $p > p_c(d)$, there exists a unique infinite open cluster \mathbb{P}_p -almost surely, and whenever $p < p_c(d)$, there is no infinite open cluster \mathbb{P}_p -almost surely. We work in the supercritical regime, and we denote the unique infinite (open) cluster by \mathbf{C}_∞ .

We may now be more specific: our goal is to explore the geometry of \mathbf{C}_∞ . There are many ways to do this, for example, one can study the asymptotic graph distance in \mathbf{C}_∞ (e.g. Antal and Pisztora [4]), one can study the asymptotic shapes of balls in the graph distance metric of \mathbf{C}_∞ (e.g. Cox and Durrett [26]), and one can study the effective resistance of \mathbf{C}_∞ within a large box (e.g. Grimmett and Kesten [38]). We study the isoperimetry of \mathbf{C}_∞ through the Cheeger constant.

By definition, $\Phi_G = 0$ for any amenable graph, and one can show that $\Phi_{\mathbf{C}_\infty} = 0$ almost surely. We will instead study the Cheeger constant of the infinite cluster in a large box, $\mathbf{C}_n := \mathbf{C}_\infty \cap [-n, n]^d$. It is known (Benjamini and Mossel [5], Mathieu and Remy [45], Rau [54], Berger, Biskup, Hoffman and Kozma [7] and Pete [49]) that $\Phi_{\mathbf{C}_n} \asymp n^{-1}$ as $n \rightarrow \infty$, prompting the following conjecture of Benjamini.

Conjecture 1.1. *For $p > p_c(d)$ and $d \geq 2$, the limit*

$$\lim_{n \rightarrow \infty} n \Phi_{\mathbf{C}_n}$$

exists \mathbb{P}_p -almost surely.

Procaccia and Rosenthal [53] made progress towards resolving this conjecture in proving upper bounds on the variance of the Cheeger constant, showing $\text{Var}(\Phi_{\mathbf{C}_n}) \leq cn^{2-d}$ for $c = c(p, d)$. Recently, Biskup, Louidor, Procaccia and Rosenthal (henceforth [BLPR]) [8] settled this conjecture positively for a natural modification of $\Phi_{\mathbf{C}_n}$ in the case $d = 2$. We define the *modified Cheeger constant* $\widehat{\Phi}_n$ of \mathbf{C}_n for dimension

$d \geq 2$ to be

$$(1.2) \quad \widehat{\Phi}_n := \min \left\{ \frac{|\partial_{\mathbf{C}_\infty} H|}{|H|} : H \subset \mathbf{C}_n, 0 < |H| \leq |\mathbf{C}_n|/d! \right\}$$

where $\partial_{\mathbf{C}_\infty} H$ denotes the open edge boundary of H within *all* of \mathbf{C}_∞ as opposed to \mathbf{C}_n . This modification is natural in the sense that a candidate subgraph H is treated as living within \mathbf{C}_∞ , and the $d!$ in the upper volume bound ensures that H need not touch the boundary of the box. Both $\Phi_{\mathbf{C}_n}$ and $\widehat{\Phi}_n$ are closely related to the so-called *anchored isoperimetric profile*, defined in the context of the infinite cluster as

$$(1.3) \quad \Phi_{\mathbf{C}_\infty,0}(n) := \inf \left\{ \frac{|\partial_{\mathbf{C}_\infty} H|}{|H|} : 0 \in H \subset \mathbf{C}_\infty, H \text{ connected}, 0 < |H| \leq n \right\}$$

where of course we must condition on the positive probability event $\{0 \in \mathbf{C}_\infty\}$. The work of [BLPR] also proved the analogue of Benjamini's conjecture for the anchored isoperimetric profile. Moreover, the subgraphs of \mathbf{C}_n and \mathbf{C}_∞ achieving each minimum were studied in both cases, and in fact were shown to scale uniformly to the same deterministic limit shape. This latter "shape theorem" implies the existence of the limit in Conjecture 1.1 for (1.2), indeed, the perimeter of this limit shape appears in the limiting value of the modified Cheeger constant.

1.2. Results. We extend the work of [BLPR] to the setting $d \geq 3$, settling Benjamini's conjecture for the modified Cheeger constant by proving a shape theorem. As the arguments of [BLPR] rely heavily on planar geometry and graph duality, a much different approach is needed. Nevertheless, we share a common starting point in the Wulff construction, described below, and there are similarities between the overall structure of the argument presented here and the argument of [BLPR]. We state the main theorem of the paper first. For each n , let \mathcal{G}_n be the (random) collection of subgraphs of \mathbf{C}_n which realize the minimum $\widehat{\Phi}_n$. For $A \subset \mathbb{R}^d$, $r > 0$ and $x \in \mathbb{R}^d$ the sets rA and $x + A$ are defined as usual by

$$(1.4) \quad rA := \{ra : a \in A\} \quad x + A := \{x + a : a \in A\}$$

and we write $\|\cdot\|_{\ell^1}$ to denote the ℓ^1 -norm of a function on \mathbb{Z}^d .

Theorem 1.2. *Let $d \geq 3$ and $p > p_c(d)$. There exists a convex set $W_{p,d} \subset [-1, 1]^d$ such that*

$$\max_{G_n \in \mathcal{G}_n} \inf_{x \in \mathbb{R}^d} n^{-d} \|\mathbf{1}_{G_n} - \mathbf{1}_{\mathbf{C}_n \cap (x + nW_{p,d})}\|_{\ell^1} \xrightarrow{n \rightarrow \infty} 0$$

holds \mathbb{P}_p -almost surely.

Following [BLPR], we build the limit shape $W_{p,d}$ through what is known as the Wulff construction, first introduced by Wulff [65] in 1901 as a possible solution to anisotropic isoperimetric problems. Given a norm τ on \mathbb{R}^d , one can form the associated isoperimetric problem, which we state in the Lipschitz setting:

$$(1.5) \quad \text{minimize } \frac{\mathcal{I}_\tau(E)}{\mathcal{L}^d(E)} \quad \text{subject to } \mathcal{L}^d(E) \leq 1$$

where the minimum runs over $E \subset \mathbb{R}^d$ with Lipschitz boundary, where \mathcal{L}^d denotes d dimensional Lebesgue measure, and where $\mathcal{I}_\tau(E)$ is defined as

$$(1.6) \quad \mathcal{I}_\tau(E) := \int_{\partial E} \tau(v_E(x)) d\mathcal{H}^{d-1}(x)$$

Here \mathcal{H}^{d-1} is the $(d-1)$ -dimensional Hausdorff measure on ∂E and $v_E(x)$ the unit exterior normal to E at the point $x \in \partial E$. Wulff's candidate isoperimetric set is constructed as follows

$$(1.7) \quad \widehat{W}_\tau := \bigcap_{v \in \mathbb{S}^{d-1}} \{x \in \mathbb{R}^d : x \cdot v \leq \tau(v)\}$$

We call \widehat{W}_τ the *unit Wulff crystal* associated to τ ; this object is the unit ball in the norm τ' dual to τ (recall that τ' is defined on $y \in \mathbb{R}^d$ by $\tau'(y) = \sup\{x \cdot y : x \in \mathbb{R}^d, \tau(x) \leq 1\}$). When \widehat{W}_τ is scaled to have unit volume,

it becomes a candidate for (1.5), and it was Taylor who ultimately proved that this rescaled shape is optimal in [59] within a wide class of Borel sets, and moreover that this rescaled shape is the unique optimizer up to translations and modifications on a null set [60].

In the following subsection, and at the beginning of Section 3, we will observe that a norm naturally emerges when our problem is viewed in the correct context. This norm, denoted $\beta_{p,d}$, is defined in a given direction by first rotating a large cube so that its top and bottom faces are normal to the given direction. We intersect this cube with \mathbb{Z}^d , and the percolation configuration on \mathbb{Z}^d restricts naturally to the discretization of the cube. We then consider the random size of a minimal cut separating the top and bottom faces of the cube. We require these cuts to be anchored near the middle of the cube, so that after taking expectations and dividing by the area of a face of the cube, we may employ a subadditivity argument to extract a limit as the diameter of the cube tends to infinity. This limit is the value of $\beta_{p,d}$ in the given direction.

We construct $\beta_{p,d}$ in Section 3, and we define the *Wulff crystal* $W_{p,d}$ to be the dilate of the unit Wulff crystal $\widehat{W}_{p,d}$ associated via (1.7) to $\beta_{p,d}$ so that $\mathcal{L}^d(W_{p,d}) = 2^d/d!$. We emphasize that the Wulff crystal is the limit shape from Theorem 1.2, and we note that the norm $\beta_{p,d}$ gives rise to a functional of the form (1.6) which we write as $\mathcal{I}_{p,d}$ and which we refer to as the *surface energy*. As in the case of [BLPR], the shape theorem we present is intimately linked with the limiting value of the Cheeger constant. Let $\theta_p(d) := \mathbb{P}_p(0 \in \mathbf{C}_\infty)$ be the density of the infinite cluster within \mathbb{Z}^d .

Theorem 1.3. *Let $d \geq 3$, $p > p_c(d)$ and let $\beta_{p,d}$ be the norm defined in Proposition 3.2. Let $W_{p,d}$ be the Wulff crystal for this norm, that is, the ball in the dual norm $\beta'_{p,d}$ such that $\mathcal{L}^d(W_{p,d}) = 2^d/d!$. Then,*

$$\lim_{n \rightarrow \infty} n \widehat{\Phi}_n = \frac{\mathcal{I}_{p,d}(W_{p,d})}{\theta_p(d) \mathcal{L}^d(W_{p,d})}$$

holds \mathbb{P}_p -almost surely.

1.3. History and discussion. Within the last thirty years, the Wulff construction has grown into an important tool in the rigorous analysis of equilibrium crystal shapes. Such problems are concerned with understanding the macroscopic behavior of one phase of matter immersed within another, respectively the *crystal* and *medium*. Careful study of the energy required to form a flat interface between the two phases in a given direction gives rise to a functional of the form (1.6), called the surface energy. It was Gibbs [35] who postulated that the asymptotic shape of the crystal should minimize this energy. The Wulff construction furnishes this minimal shape.

We work almost exclusively within \mathbf{C}_∞ , so it is natural to wonder where our problem of characterizing the shape of Cheeger optimizers fits into the above paradigm. We regard each Cheeger optimizer G_n as a large droplet of the crystalline phase, and we regard the complementary graph in \mathbf{C}_∞ as a medium. In this way, it is possible to construct a surface energy associated to Cheeger optimizers, and this is exactly the motivation for the definition of $\beta_{p,d}$.

The spirit of Theorem 1.2 can be traced back to the work of Milnos and Sinai [46, 47] from the 1960s, in which the geometric properties of phase separation in a material are rigorously studied. The first rigorous characterization of phase separation via the Wulff construction was given by Dobrushin, Kotecký and Shlosman (henceforth [DKS]) [29] in the context of the two dimensional Ising model. Independently, and at around the same time, the Wulff construction was used in the context of two dimensional bond percolation by Alexander, Chayes and Chayes [2] to characterize the asymptotic shape of a finite open cluster in the supercritical regime. The two dimensional results of [DKS], valid for the low temperature regime, were extended up to the critical temperature thanks to the work of Ioffe in [39] and Ioffe and Schonmann [40].

The first rigorous derivation of the Wulff construction for a genuine short-range model in three dimensions was achieved by Cerf in the context of bond percolation [17]. This result was then extended to the Ising model and to higher dimensions by several substantial works of Bodineau [10, 11] and Cerf and Pisztora [19, 20]. The coarse graining results of Pisztora [52] played an integral role in this study of the Ising model, FK percolation and bond percolation in higher dimensions. A comprehensive survey of these results and of

others can be found in Section 5.5 of Cerf's monograph [18] and in the review article of Bodineau, Ioffe and Velenik [13].

In all cases, the jump to dimensions $d > 2$ has, at least so far, necessitated a shift from the uniform topology to the ℓ^1 topology on the space of shapes (we are intentionally vague about which space we consider). Indeed, as noted by [DKS] in [29], the variational problem (1.5) is not stable in $d \geq 3$ when the space of shapes is equipped with the uniform topology. That is, in $d \geq 3$, it is possible to construct sequence of shapes which are bounded away from the optimal shape in the uniform topology, but whose surface energies tend to the optimal surface energy. This has implications at the microscopic level; if one desired to prove a uniform shape theorem in $d \geq 3$ for the Cheeger optimizers, one would first have to rule out the existence of long but thin filaments (as in Figure 1) in these discrete objects with high probability.

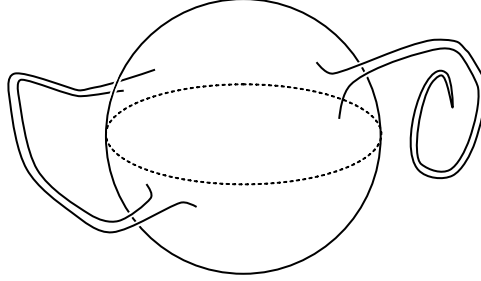


FIGURE 1. In $d = 3$, filaments added to the optimal shape for the Euclidean isoperimetric problem produce a set which is almost optimal and which has large uniform distance to the sphere.

This lack of regularity at the microscopic level requires that we consider the variational problem over a wider class of shapes, and it is here that geometric measure theory emerges as a valuable tool (as first realized by Alberti, Bellettini, Cassandro and Presutti [1]).

1.4. Outline. We proceed with an outline of our strategy. Our goals may be summarized as follows: we wish to show that the sequence of discrete, random isoperimetric problems (1.2) scale to a continuous, deterministic isoperimetric problem (1.5) corresponding to some norm $\beta_{p,d}$ on \mathbb{R}^d . This could be phrased in terms of Γ -convergence, and indeed, this language has been used in some of the results outlined in our historical discussion, as well as in recent related work of Braides and Piatnitski [14, 15].

The first task is to construct a suitable norm $\beta_{p,d}$ on \mathbb{R}^d . This is done in Section 3 after introducing some definitions and notation in Section 2. The key to the existence of $\beta_{p,d}$ is a spatial subadditivity argument applied to the geometric setting described briefly before Theorem 1.3.

The resulting norm $\beta_{p,d}$ gives rise to a surface energy $\mathcal{I}_{p,d}$, and the remainder of the paper is concerned with demonstrating that the unique optimizer of the isoperimetric problem associated to $\mathcal{I}_{p,d}$ faithfully describes the macroscopic shape of each large $G_n \in \mathcal{G}_n$. We must show a strong correspondence between discrete objects (the various subgraphs of \mathbf{C}_n) and continuous objects (Borel subsets of $[-1, 1]^d$ for which isoperimetric problems can be defined). Specifically, this correspondence should link the isoperimetric ratio of subgraphs of \mathbf{C}_n to the corresponding ratio for continuous objects, as in the limiting value of Theorem 1.3.

In order to pass from continuous objects to discrete objects, we must first prove concentration estimates for the random variables used to define $\beta_{p,d}$. This is in line with the strategy of [BLPR], and is in contrast to large deviation methods used in the papers on the Wulff construction mentioned in Section 1.3. We prove these concentration estimates in Section 4.

Consequences of these concentration estimates are presented in Section 5, where we use the results of Section 4 to give a high probability upper bound on $n\Phi_n$. To obtain this upper bound, we intersect a large polytope P with \mathbf{C}_n to produce a subgraph H_n of \mathbf{C}_n . We exhibit control on the volume of H_n which ensures that H_n meets the criteria of (1.2). Our concentration estimates relate the size of the open edge boundary

of H_n to the surface energy of the original polytope. At the end of Section 5, we will have proved half of Theorem 1.3. All of the arguments presented up to this point work in the setting $d \geq 2$.

Passing from discrete objects to continuous objects is significantly more delicate. The main difficulty is to construct a suitable continuum object P_n (for instance, a polytope in $[-1, 1]^d$) for each $G_n \in \mathcal{G}_n$ so that the perimeter of P_n stays bounded as $n \rightarrow \infty$. This is clearly necessary if we are trying to relate the isoperimetric problems in the discrete setting to (1.5) in the continuum for the norm $\beta_{p,d}$. A natural first guess for P_n is the union of unit cubes centered at all vertices of G_n . However, due to percolation of closed edges near $p_c(d)$, we do not have control on the perimeters of such P_n unless p is very close to one.

This suggests a renormalization argument, which we introduce in Section 6. We base our argument on a construction due to Zhang from [67], but we must still modify this construction and study it carefully in order to apply it to our situation. It is here that, for reasons which will be made clear in Section 6, we must restrict ourselves to the setting $d \geq 3$. This is no loss as the case $d = 2$ is covered by the results of [BLPR] in [8].

In Section 7, we will reap the efforts of Section 6, passing from $G_n \in \mathcal{G}_n$ to sets of finite perimeter (defined in Section 2). Such sets have sufficient regularity that we may work locally on their boundaries. This is instrumental to Section 8, in which we relate the size of the open edge boundary of each G_n to the surface energy of their scaling limit. The results from Section 5 will then tell us that each G_n must be close to the Wulff crystal with high probability, thus giving Theorem 1.2 and Theorem 1.3 in quick succession.

1.5. Open problems. We now pose some natural open questions, some of which were originally stated by [BLPR] in [8].

(1) *Free boundary conditions and more general domains:* Conjecture 1.1 is still open for the unmodified Cheeger constant. We expect that it is possible to adapt the approach of [BLPR] in [8] to resolve this in $d = 2$. It is not entirely obvious what the limit shape should be in this case, or even whether it is unique. One conjecture is that an optimal shape will be a rescaled quarter-Wulff crystal in one of the corners of the square. This conjecture is motivated by the Winterbottom construction introduced in [64], which is an analogue of the Wulff construction for crystals in the presence of a wall. This construction has been used successfully in the two dimensional Ising model by Pfister and Velenik [50, 51] and in higher dimensions by Bodineau, Ioffe and Velenik [12].

One can generalize Benajmini's conjecture in the two dimensional case to domains other than boxes; given a nice bounded open set $\Omega \subset \mathbb{R}^2$, one can study the asymptotics of the unmodified Cheeger constant as well as the shapes of the Cheeger optimizers for $\mathbf{C}_\infty \cap n\Omega$. Results characterizing the limiting value of the Cheeger constant or the limiting shapes of the optimizers would be isoperimetric analogues of the work of Cerf and Th  ret [23] on minimal cutsets (in $d \geq 2$).

(2) *More information on the Wulff crystal:* Little is known about the geometric properties of the Wulff crystal. One recent result of Garet, Marchand, Procaccia and Th  ret [34] is that, in $d = 2$, the Wulff crystal varies continuously with respect to the uniform metric on compact sets as a function of the percolation parameter $p \in (p_c(2), 1]$. It was conjectured in [8] that the two dimensional Wulff crystal tends to a Euclidean ball as $p \downarrow p_c(2)$; this is still widely open. It is natural to wonder whether the Wulff crystal has facets (open portions of the boundary with zero curvature), corners and how such questions depend on the percolation parameter.

(3) *Uniform convergence for $d \geq 3$:* An interesting and challenging question is whether a form of Theorem 1.2 holds in $d \geq 3$ when we replace ℓ^1 convergence by uniform convergence. Despite the complications described above, [DKS] expressed optimism in [29] about the ability to rule out the existence of filaments in $d \geq 3$ in the context of the Ising model.

1.6. Acknowledgements. I thank my advisor Marek Biskup for suggesting this problem, for his guidance and his support. I am deeply indebted to Rapha  l Cerf for patiently sharing his expertise and insight during my time in Paris. I likewise thank Eviatar Procaccia for his guidance. I am grateful to Vincent Vargas,

Claire Berenger and Shannon Starr for making it possible for me to attend the IHP Disordered Systems trimester. Finally, I thank Yoshihiro Abe, Ian Charlesworth, Arko Chatterjee, Hugo Duminil-Copin, Aukosh Jagannath, Ben Krause, Sangchul Lee, Tom Liggett, Jeff Lin, Peter Petersen, Jacob Rooney and Ian Zemke for helpful conversations. This research has been partially supported by the NSF grant DMS-1407558.

2. DEFINITIONS AND NOTATION

2.1. Graphs. Throughout this paper, we work within the graph \mathbb{Z}^d . The vertex set $V(\mathbb{Z}^d)$ of \mathbb{Z}^d is the set of integer d -tuples. There is an edge between two d -tuples if, when viewed as vectors in \mathbb{R}^d , they have Euclidean distance one, and we denote the edge set of \mathbb{Z}^d as $E(\mathbb{Z}^d)$. If $x, y \in \mathbb{Z}^d$ share an edge, we say they are adjacent vertices and we write $x \sim y$. Given two vertices $x, y \in \mathbb{Z}^d$, a *path from x to y* is a finite, alternating sequence of vertices and edges $x_0, e_1, x_1, \dots, e_m, x_m$ such that e_i is the edge shared by x_{i-1} and x_i for $i = 1, \dots, m$, and such that $x_0 = x$ and $x_m = y$. We say the path *joins* the vertices x and y , and the *length* of the path is the number of edges m in this sequence. A subgraph $G \subset \mathbb{Z}^d$ is *connected* if for any two vertices $x, y \in G$, there is a path using only vertices and edges in G which joins x and y .

Given $x \in \mathbb{Z}^d$, a *path from x to ∞* is an infinite alternating sequence of vertices and edges x_0, e_1, x_1, \dots such that $x = x_0$ and such that no finite box contains all edges in this path. A path is said to be *simple* if it does not use any vertex more than once. In either case, we often regard paths as sequences of edges out of convenience. We need several notions of the boundary of a subgraph of \mathbb{Z}^d . Given G a finite subgraph of \mathbb{Z}^d , we define the *edge boundary* and *outer edge boundary* of G to respectively be

$$(2.1) \quad \partial G := \{e \in \mathbb{Z}^d : \text{exactly one endpoint of } e \text{ lies in } G\}$$

$$(2.2) \quad \partial_o G := \left\{ e \in \partial G : \begin{array}{l} \text{the endpoint of } e \text{ in } G \text{ is connected to } \infty \\ \text{via a path which uses no other vertices of } G \end{array} \right\}$$

We also define the *outer vertex boundary* of G to be

$$(2.3) \quad \partial_* G := \{v \in G : v \text{ is an endpoint of an edge in } \partial_o G\}$$

Given a finite subgraph $G \subset \mathbb{Z}^d$, a *cutset separating G from ∞* is a finite collection of edges $S \subset E(\mathbb{Z}^d)$ such any path from a vertex of G to ∞ must use an edge in the set S . If $A, B \subset V(G)$ are disjoint vertex sets, a *cutset separating A and B* is a finite collection of edges $S \subset E(G)$ such that any path from a vertex of A to a vertex of B must use an edge of S . In either case, a cutset S is said to be *minimal* if it is no longer a cutset upon removing any edge in S .

We define \mathbb{L}^d to be the graph with vertex set $V(\mathbb{Z}^d)$ and edge set consisting of pairs of vertices $x, y \in \mathbb{Z}^d$ which, when viewed as vectors in \mathbb{R}^d , have ℓ^∞ -distance one. If $x, y \in \mathbb{Z}^d$ are joined by an edge in \mathbb{L}^d , we say the two vertices are \mathbb{L}^d -adjacent and write $x \sim_{\mathbb{L}} y$. We define \mathbb{L}^d -paths analogously to paths in \mathbb{Z}^d , and we say that a subgraph $G \subset \mathbb{Z}^d$ is \mathbb{L}^d -connected if any two vertices $x, y \in G$ are joined by an \mathbb{L}^d -path. The following proposition, due to Deuschel and Pisztora [27] (Lemma 2.1), provides a link between cutsets in \mathbb{Z}^d and the notion of \mathbb{L}^d -connectivity. More recently, Timár has given a concise and more combinatorial proof of a stronger statement in [63].

Proposition 2.1. *Let $G \subset \mathbb{Z}^d$ be a finite, connected subgraph of \mathbb{Z}^d . Then the outer vertex boundary $\partial_* G$ is \mathbb{L}^d -connected, as is the set of endpoint vertices of edges in $\partial_o G$.*

Such results are important to the execution of Peierls estimates (used in conjunction with a statement like Proposition A.12) appearing frequently in percolation and other lattice models. Finally, we mention that for G a subgraph of \mathbb{Z}^d and $K \subset \mathbb{R}^d$ compact, we will often write $G \cap K$ as shorthand for $V(G) \cap K$. The vertex set $V(G) \cap K$ inherits a graph structure from G , so that when referring to $e \in G \cap K$ for an edge e , it is understood that both endpoints of e lie in $V(G) \cap K$. Finally, if G is a finite set, we use $|G|$ to denote the cardinality of G . If G is a finite subgraph of \mathbb{Z}^d , we write $|G|$ in place of the cardinality of $V(G)$.

2.2. Percolation. The probabilistic setting of this paper is bond percolation on \mathbb{Z}^d with $d \geq 2$. Percolation gives rise to another notion of graph boundary: if G is a finite subgraph of \mathbf{C}_∞ , we define the *open edge boundary* of G as

$$(2.4) \quad \partial^\omega G := \{e \in \partial G : \omega(e) = 1\}$$

Recall that \mathbf{C}_n was defined to be $\mathbf{C}_\infty \cap [-n, n]^d$. Given a subgraph G of \mathbf{C}_∞ , the ratio $|\partial^\omega G|/|G|$ shall be called the *conductance* of G and written as φ_G . This is consistent with the terminology used in [48], for instance. We may rewrite the definition of the modified Cheeger constant as

$$(2.5) \quad \widehat{\Phi}_n := \min \left\{ \varphi_G : G \subset \mathbf{C}_n, 0 < |G| \leq \frac{|\mathbf{C}_n|}{d!} \right\}$$

We'll say a subgraph G of \mathbf{C}_n is *valid* if it satisfies $0 < |G| \leq (|\mathbf{C}_n|/d!)$. A valid subgraph $G \subset \mathbf{C}_n$ is *optimal* if $\varphi_G = \widehat{\Phi}_n$, and we let \mathcal{G}_n denote the collection of all optimal subgraphs of \mathbf{C}_n . We observe that each element of \mathcal{G}_n is determined by its vertex set, in the sense that each $G_n \in \mathcal{G}_n$ inherits its graph structure from \mathbf{C}_n . If this were not the case for some G_n , we could strictly reduce its open edge boundary.

2.3. A preliminary setting of convergence. To prove Theorem 1.2, we will first encode each optimizer G_n as a measure and prove convergence to a limiting measure. Given $G_n \in \mathcal{G}_n$, we define the *empirical measure* of G_n as

$$(2.6) \quad \mu_n := \frac{1}{n^d} \sum_{x \in G_n} \delta_{x/n}$$

where the sum ranges over all vertices of G_n , so that each μ_n is a random, non-negative Borel measure on $[-1, 1]^d$. Given a Borel set $E \subset [-1, 1]^d$, we define ν_E as the measure on $[-1, 1]^d$ having density $\theta_p(d)\mathbf{1}_E$ with respect to Lebesgue measure, and we say that ν_E *represents* the set E . The collection of signed Borel measures on $[-1, 1]^d$ shall be denoted as $\mathcal{M}([-1, 1]^d)$, and the closed weak ball of radius 3^d about the zero measure in this space shall be written as \mathcal{B}_d . We observe that for each configuration ω , the empirical measures μ_n lie within \mathcal{B}_d , as do the representative measures of every Borel set $E \subset [-1, 1]^d$. We equip \mathcal{B}_d with a metric \mathfrak{d} defined as follows.

For $k \in \{0, 1, 2, \dots\}$, let Δ^k denote the collection of closed dyadic cubes in $[-1, 1]^d$ at scale k . Specifically, each cube is a translate of $[-2^{-k}, 2^{-k}]^d$. Given $\mu, \nu \in \mathcal{B}_d$, we define

$$(2.7) \quad \mathfrak{d}(\mu, \nu) := \sum_{k=0}^{\infty} \frac{1}{2^k} \sum_{Q \in \Delta^k} \frac{1}{|\Delta^k|} |\mu(Q) - \nu(Q)|$$

Observe that \mathfrak{d} encodes weak* convergence of measures in \mathcal{B}_d . As a precursor to Theorem 1.2, we will prove the following theorem first in Section 8.

Theorem 2.2. *For $d \geq 3$ and $p > p_c(d)$, let $W_{p,d}$ be the Wulff crystal from Theorem 1.2. Define the following subset of $\mathcal{M}([-1, 1]^d)$*

$$\mathcal{W}_{p,d} := \{\nu_E : E = W_{p,d} + x, \text{ with } W_{p,d} + x \subset [-1, 1]^d\}$$

We have that \mathbb{P}_p -almost surely,

$$\max_{G_n \in \mathcal{G}_n} \mathfrak{d}(\mu_n, \mathcal{W}_{p,d}) \xrightarrow{n \rightarrow \infty} 0$$

2.4. Some geometric measure theory. Throughout the paper, we write \mathcal{L}^d to denote d dimensional Lebesgue measure and \mathcal{H}^d to denote d dimensional Hausdorff measure. We now introduce sets of finite perimeter; the following definitions are taken from Sections 13.3 and 14.1 of [18].

For $E \subset \mathbb{R}^d$ a Borel set and $O \subset \mathbb{R}^d$ open, we define the *perimeter of E in O* to be

$$(2.8) \quad \text{per}(E, O) := \sup \left\{ \int_E \text{div} f(x) d\mathcal{L}^d(x) : f \in C_c^\infty(O, B(0, 1)) \right\}$$

where $B(0, 1)$ denotes the Euclidean unit ball in \mathbb{R}^d . We write $\text{per}(E, \mathbb{R}^d)$ as $\text{per}(E)$ and say that $\text{per}(E)$ is the *perimeter* of E . A Borel set $E \subset \mathbb{R}^d$ has *finite perimeter* if $\text{per}(E) < \infty$, and has *locally finite perimeter* if $\text{per}(E, O) < \infty$ for each bounded open O . Sets of locally finite perimeter are also known as *Caccioppoli sets*.

We can generalize the definition (2.8) to other norms τ on \mathbb{R}^d by using the unit Wulff crystal for τ (1.7) in place of $B(0, 1)$. This extends the definition of the surface energy (1.6) to Borel sets. For a Borel set $E \subset \mathbb{R}^d$ and O open, we define the *surface energy of E in O with respect to τ* as

$$(2.9) \quad \mathcal{I}_\tau(E, O) = \sup \left\{ \int_E \text{div} f(x) d\mathcal{L}^d(x) : f \in C_c^\infty(O, \widehat{W}_\tau) \right\}$$

and we write $\mathcal{I}_\tau(E, \mathbb{R}^d)$ as $\mathcal{I}_\tau(E)$. It is a consequence of the divergence theorem that (2.9) is consistent with (1.6). We now formally state the following important theorem due to Taylor [59, 60, 61] mentioned in the introduction.

Theorem 2.3. *Let τ be a norm on \mathbb{R}^d and consider the variational problem for Borel sets $E \subset \mathbb{R}^d$:*

$$\text{minimize } \mathcal{I}_\tau(E) \text{ subject to } \mathcal{L}^d(E) \geq \mathcal{L}^d(W_\tau)$$

A set E is a solution to this variational problem if and only if there exists $x \in \mathbb{R}^d$ such that $W_\tau \Delta (E + x)$ has Lebesgue measure zero. That is, up to modifications on null sets and translation, the Wulff crystal is the unique minimizer of this variational problem.

We relegate other results concerning sets of finite perimeter and surface energy to the appendix.

3. THE NORM $\beta_{p,d}$ AND THE WULFF CRYSTAL

We now introduce objects fundamental to defining the norm $\beta_{p,d}$ and hence the Wulff crystal $W_{p,d}$. To motivate our construction, we appeal to the following heuristic: we regard an optimizer $G_n \in \mathcal{G}_n$ as a droplet in \mathbb{C}_∞ , and we look at a small but macroscopic (diameter on the order of n) box intersecting the boundary of G_n , as depicted in Figure 2.

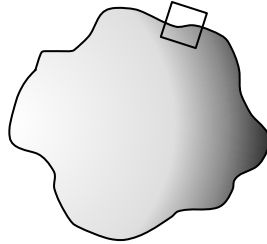


FIGURE 2. A small macroscopic box on the boundary of G_n .

Let us first discuss some of the assumptions implicitly made by Figure 2: the G_n are discrete objects, and the representation of G_n above treats ∂G_n as though it is a continuous object. This can be justified by appealing to Proposition 2.1, in which \mathbb{L}^d -connectivity of $\partial_o G_n$ can be thought of as a discrete substitute for

continuity. Accepting that the boundary of G_n behaves like a continuous object macroscopically, we note that G_n may not even be connected, simply connected or otherwise may have various holes.

Nonetheless, let us proceed with our heuristic: the small box B captures a piece of ∂G_n , and one can imagine this portion of ∂G_n separating the top and bottom faces of B . Even this makes assumptions on the “regularity” of the ∂G_n , as it may be that ∂G_n has small spikes which shoot out of B , preventing $\partial G_n \cap B$ from truly separating the top and bottom faces of B . We will see in Section 8 that for boxes (or other objects) chosen well relative to G_n , we can arrange that $\partial G_n \cap B$ almost separates the top and bottom faces of B , meaning that we can produce a cutset from $\partial G_n \cap B$ by adding only a few more edges.

As G_n is a Cheeger optimizer, if $\partial G_n \cap B$ separates the top and bottom faces of B , this cutset should use the fewest number of open edges possible in order to minimize $\partial^\omega G_n$. The choice of this cutset should not greatly affect the enclosed volume $|G_n|$ because B is so small relative to G_n , so we imagine that optimizing the number of open edges used by this cut is most important to minimizing the conductance φ_{G_n} of G_n . The minimal number of open edges used by a cut separating the top and bottom faces of B acts as the surface energy in the direction normal to these faces. We expect this energy to grow with n as n^{d-1} regardless the direction v , so our strategy is to construct $\beta_{p,d}$ as a limit of these discrete surface energies normalized by a factor of n^{d-1} . For the purpose of implementing a subadditivity argument, it will be important that these cuts are “anchored” in some way to the equator of the box. We will make this notion clear in Section 3.1.

In two dimensions, the dual edges to any cutset form a path, so studying the minimal random weight along all such cutsets falls under the umbrella of first passage percolation. We are fortunate that minimal randomly weighted cutsets in boxes for dimensions $d \geq 3$ are also well studied objects. They were first studied by Kesten in [42] as a means of studying a higher-dimensional version of first passage percolation (where the dual squares to each edge in a cut form a surface with random weights). Since this time, Th  ret [62], Rossignol and Th  ret [55, 56], Zhang [67] and Garet [33] have all studied variants of this problem. For a detailed list of these results, see Section 3.1 of [23]. As mentioned in Section 1.5, Cerf and Th  ret [23] have obtained a law of large numbers for the randomly weighted cuts separating pieces of the boundary of a very general domain in d dimensions. The norm $\beta_{p,d}$ which we will soon construct has been used in most of the work just mentioned, so we emphasize that the results presented in this section and the following are not new or even the best possible. Nevertheless, we find it important to present a relatively self-contained argument, and we will use the notation introduced in this section extensively throughout the paper.

3.1. Discrete cylinders, cutsets and connectivity. We take much of our notation from the work of Cerf [18] and of Cerf and Th  ret [23]. Let $F \subset \mathbb{R}^d$ be the isometric image of either a non-degenerate polytope in \mathbb{R}^{d-1} or a Euclidean ball in \mathbb{R}^{d-1} . We write $\text{hyp}(F)$ to denote the hyperplane spanned by F , and we let $v(F)$ denote of the two unit vectors in \mathbb{S}^{d-1} normal to $\text{hyp}(F)$. We will define exactly what we mean by polytope in Section 5; in the present section we will only ever need F to be a unit square.

For $\rho > 0$, we define $\text{cyl}(F, \rho)$ to be the closed cylinder in \mathbb{R}^d whose top and bottom faces are respectively $F_\rho^+ := F + \rho v(F)$ and $F_\rho^- := F - \rho v(F)$. The choice of $v(F)$ creates some ambiguity over which face of the cylinder is the top, but this ambiguity will be unimportant throughout the paper, and will play no role in the definition of the norm. We also define

$$(3.1) \quad \text{slab}(F, \rho) := \{y \in \text{hyp}(A) + rv : r \in [-\rho, \rho]\}$$

We now proceed to discretize $\text{cyl}(F, \rho)$ and portions of its boundary. More specifically, we perform these discretization on dilates of $\text{cyl}(F, \rho)$, so that they become progressively more faithful to the sets from which they arise as the dilation parameter grows. Define the discrete cylinder $\text{d-cyl}(F, \rho, r)$ as follows.

$$(3.2) \quad \text{d-cyl}(F, \rho, r) := (r\text{cyl}(F, \rho)) \cap \mathbb{Z}^d$$

Note that $\text{cyl}(F, \rho) \setminus \text{hyp}(F)$ consists of two connected components. We will denote the top component by $\text{cyl}^+(F, \rho)$, this is the component containing F_ρ^+ . Likewise, the bottom component is the one containing F_ρ^- and shall be denoted $\text{cyl}^-(F, \rho)$. The following sets of vertices are the top (corresponding to “+”) and

bottom (“−”) *hemispheres* of $\text{d-cyl}(F, \rho, r)$:

$$(3.3) \quad \text{d-hemi}^\pm(F, \rho, r) := \{y \in \partial_* \text{d-cyl}(F, \rho, r) : y \in \text{rcyl}^\pm(F, \rho)\}$$

We also define the top and bottom *faces* of $\text{d-cyl}(F, \rho, r)$:

$$(3.4) \quad \text{d-face}^\pm(F, \rho, r) := \{y \in \partial_*[(\text{rslab}(F, \rho)) \cap \mathbb{Z}^d] : y \in \text{rcyl}^\pm(F, \rho, r)\}$$

Note that the vertex sets $\text{d-hemi}^\pm(F, \rho, r)$ and $\text{d-face}^\pm(F, \rho, r)$ are contained in $\text{d-cyl}(F, \rho, r)$. The latter vertex set $\text{d-cyl}(F, \rho, r)$ inherits a graph structure from \mathbb{Z}^d , so we may consider cutsets S separating the vertex sets $\text{d-hemi}^\pm(F, \rho, r)$ or alternatively, cutsets separating $\text{d-face}^\pm(F, \rho, r)$. Given a cutset S of this form, we let $|S|$ denote the cardinality of S . However, bond percolation on \mathbb{Z}^d induces a bond percolation on $\text{d-cyl}(F, \rho, r)$, so to any cutset S of the form just described we may also assign a value $|S|_\omega$, the number of open edges in S . For any fixed cutset S , it follows that $|S|_\omega$ is a random variable.

We define the random variable $\mathfrak{X}_{\text{hemi}}(F, \rho, r)$ to be the minimum of $|S|_\omega$, where S ranges over all cutsets separating $\text{d-hemi}^\pm(F, \rho, r)$ within $\text{d-cyl}(F, \rho, r)$. Likewise, we define $\mathfrak{X}_{\text{face}}(F, \rho, r)$ to be the minimum of $|S|_\omega$, with S ranging over all cutsets separating $\text{d-face}^\pm(F, \rho, r)$. In either case, we may restrict the minimum to one taken over all minimal cutsets. Figure 3 is a representation of the cuts we consider in either case. Having set our notation, we are ready to construct $\beta_{p,d}$.

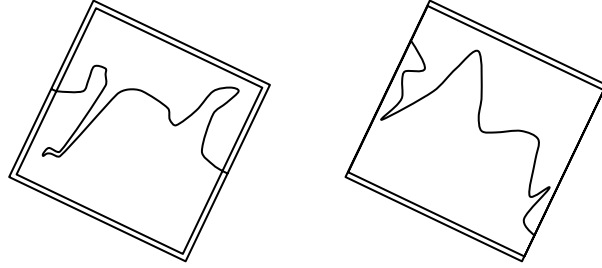


FIGURE 3. On the left, cutsets separating $\text{d-hemi}^\pm(F, \rho, r)$ are anchored at the “equator” of the cylinder $\text{rcyl}(F, \rho)$. This is in contrast to cutsets separating $\text{d-face}^\pm(F, \rho, r)$, depicted on the right.

3.2. Defining the norm. The norm $\beta_{p,d}$ will be built from cylinders over $(d-1)$ -cubes or *squares*. For $x \in \mathbb{R}^d$ and $v \in \mathbb{S}^{d-1}$, we consider an isometric image of the square $[-1, 1]^{d-1} \times \{0\}$, translated in \mathbb{R}^d so that it is centered at x , and rotated so that the hyperplane it spans is normal to v . We denote this square as $S(x, v)$. For $d \geq 3$, these conditions do not determine $S(x, v)$ uniquely, as we can rotate one choice to obtain many others. In order to make our definitions concrete, we will need to choose one such rotation for each $v \in \mathbb{S}^{d-1}$.

Formally, this is a function f from \mathbb{S}^{d-1} to the set of unit $(d-1)$ -squares (isometric images of $[-1, 1]^{d-1} \times \{0\}$) such that the square $f(v) = S(0, v)$ spans the hyperplane orthogonal to v . We choose one such function f and refer to it as the *chosen rotation*. We make the chosen rotation explicit in the next subsection, as we will need to arrange that f varies in a nice way over most of the sphere. We state our assumptions about the chosen rotation later, as they will not affect the proofs in this section, and indeed we will show in Proposition 3.2 that the value of $\beta_{p,d}$ does not depend on the chosen rotation.

The random variable used to define the norm is

$$(3.5) \quad \mathfrak{X}(x, v, r) := \mathfrak{X}_{\text{hemi}}(S(x, v), 1, r)$$

As was mentioned at the beginning of this section, these objects are well-studied. In particular, precise large deviation estimates are well known for $\mathfrak{X}(x, v, r)$, giving rise to a law of large numbers for $\mathfrak{X}(x, v, r)/r^d$. To define the norm as quickly as possible, we use a subadditivity argument on the expectations of these random

variables. Our argument is essentially the one given by Rossignol and Th  ret in Section 4.3 of [56]. Before carrying out this argument, we make one observation about the variables $\mathfrak{X}(x, v, r)$ first. For $a > 0$ and $E \subset \mathbb{R}^d$ Borel, we define the set $N_a(E)$ is the closed Euclidean a -neighborhood of the set E .

Lemma 3.1. *Let $x \in \mathbb{R}^d$, $v \in \mathbb{S}^{d-1}$ and $r > 0$. Then there is a positive $c(d)$ so that*

$$\mathbb{E}_p \mathfrak{X}(x, v, r + d^{1/2}) \leq \mathbb{E}_p \mathfrak{X}(0, v, r) + c(d)r^{d-2}$$

Proof. Fix ω , let $x \in \mathbb{R}^d$ and choose $x' \in \mathbb{R}^d$ so that $|x - x'|_2 \leq d^{1/2}$. Note that

$$(3.6) \quad \text{rcyl}(S(x', v), 1) \subset (r + d^{1/2})\text{cyl}(S(x, v), 1)$$

Let A_r be the collection of edges having non-empty intersection with the set

$$(3.7) \quad N := N_{5d}((r + d^{1/2})S(x, v) \setminus \text{rcyl}(S(x, v), 1))$$

It follows immediately that $|A_r| \leq c(d)r^{d-2}$. Let $E_r(\omega)$ be a minimal cutset separating the hemispheres $\text{d-hemi}^\pm(S(x', v), 1, r)$ in $\text{d-cyl}(S(x', v), 1, r)$, and choose $E_r(\omega)$ so that within the configuration ω , we have $\mathfrak{X}(x', v, r) = |E_r(\omega)|_\omega$. We claim that the collection of edges in $A_r \cup E_r(\omega)$ which lie in $\text{d-cyl}(S(x, v), 1, r + d^{1/2})$ separate the hemispheres $\text{d-hemi}^\pm(S(x, v), 1, r + d^{1/2})$ within $\text{d-cyl}(S(x, v), 1, r + d^{1/2})$. Assuming this for now, it follows that

$$(3.8) \quad \mathfrak{X}(x, v, r + \sqrt{d}) \leq \mathfrak{X}(x', v, r) + c(d)r^{d-2}$$

and the desired claim holds upon taking expectations as $x' \in \mathbb{Z}^d$. To complete the proof, it suffices to show that any \mathbb{Z}^d path joining $\text{d-hemi}^\pm(S(x, v), 1, r + d^{1/2})$ within $\text{d-cyl}(S(x, v), 1, r + d^{1/2})$ must use an edge of $A_r \cup E_r(\omega)$. We do this carefully now.

Let $y^\pm \in \text{d-hemi}^\pm(S(x, v), 1, r + d^{1/2})$, and let γ be a simple path from y^- to y^+ within the discrete cylinder $\text{d-cyl}(S(x, v), 1, r + d^{1/2})$. If γ does not pass through any vertex of $\partial_* \text{d-cyl}(S(x', v), 1, r)$, it must be that γ lies entirely within $(r + d^{1/2})\text{cyl}(S(x, v), 1) \setminus \text{rcyl}(S(x', v), 1)$, in which case γ must use an edge of A_r .

We may then suppose that γ passes through a vertex of $\partial_* \text{d-cyl}(S(x', v), 1, r)$. We now consider several cases.

Case I: Suppose that the last vertex z^+ of $\text{d-cyl}(S(x', v), 1, r)$ used by γ lies within the bottom hemisphere $\text{d-hemi}^-(S(x', v), 1, r)$. If we let γ' denote the remainder of γ started at z^+ , we observe that γ' is contained within $(r + d^{1/2})\text{cyl}(S(x, v), 1) \setminus \text{rcyl}(S(x', v), 1)$, and moreover, that γ' either begins in the bottom half of $(r + d^{1/2})\text{cyl}(S(x, v), 1)$ or in the neighborhood N defined above. In either case, γ' must use an edge in A_r .

Case II: Suppose that the first vertex z^- of $\text{d-cyl}(S(x', v), 1, r)$ used by γ lies in $\text{d-hemi}^+(S(x', v), 1, r)$. Similar reasoning shows that γ must use an edge in A_r before reaching z^- .

Case III: We may now suppose that $z^\pm \in \text{d-hemi}^\pm(S(x', v), 1, r)$. Let z denote the last vertex of $\text{d-hemi}^-(S(x', v), 1, r)$ used by γ , and consider the truncation γ' of γ which joins z to z^+ . If γ' is contained completely within $\text{d-cyl}(S(x', v), 1, r)$, it must be that γ' uses an edge of $E_r(\omega)$.

On the other hand, if γ' is not contained in $\text{d-cyl}(S(x', v), 1, r)$, we may assume that the vertex following z along the path γ' must lie outside of $\text{d-cyl}(S(x', v), 1, r)$, else γ' would either use an edge of $E_r(\omega)$, or would not use z last among all vertices of $\text{d-hemi}^-(S(x', v), 1, r)$.

With this assumption made, we see that γ' leaves $\text{d-cyl}(S(x', v), 1, r)$ at the vertex z , and only returns to $\text{d-cyl}(S(x', v), 1, r)$ at some vertex $z' \in \text{d-hemi}^+(S(x', v), 1, r)$. If we restrict γ' to the path γ'' joining z to z' , we see that all intermediate vertices lie in $(r + d^{1/2})\text{cyl}(S(x, v), 1) \setminus \text{rcyl}(S(x', v), 1)$, which shows γ'' uses an edge of A_r due to the locations of its starting and ending points. \square

With this lemma in place, we now define $\beta_{p,d}$ as a function on \mathbb{S}^{d-1} .

Proposition 3.2. *Let $d \geq 2$ and $p > p_c(d)$. For all $v \in \mathbb{S}^{d-1}$, the following limit exists, is finite and defines $\beta_{p,d}(v)$.*

$$\beta_{p,d}(v) := \lim_{n \rightarrow \infty} \frac{\mathbb{E}_p \mathfrak{X}(0, v, n)}{(2n)^{d-1}}$$

Moreover, this limit is independent of the chosen rotation for the direction v .

Proof. Let $n, m \in \mathbb{N}$ with $n > m$ and both numbers larger than d . We write $n = km + r$ for $k, r \in \mathbb{N} \cup \{0\}$ and $r < m$. Let $S(x, v)$ denote the chosen square for the direction v centered at x , and let $T(x, v)$ denote any rotation of $S(x, v)$ about the origin such that $\text{hyp}(T(x, v)) = \text{hyp}(S(x, v))$. We let $\mathfrak{Y}(x, v, r)$ abbreviate the random variable $\mathfrak{X}_{\text{hemi}}(T(x, v), 1, r)$. Choose a finite collection $\{T_i\}_{i=1}^\ell$ of translates of $(m + d^{1/2})T(0, v)$, each contained within $nS(0, v)$, so that:

- (i) These translates are disjoint.
- (ii) $\mathcal{H}^{d-1}\left(nS(0, v) \setminus \bigcup_{i=1}^\ell T_i\right) \leq c(d)mn^{d-2}$.
- (iii) $\ell \leq (k+1)^{d-1}$.

Fix a configuration ω , and for each T_i , let $E_i(\omega)$ be a minimal cutset in the configuration ω separating the hemispheres $\text{d-hemi}^\pm(T_i, m + d^{1/2}, 1)$ within $\text{d-cyl}(T_i, m + d^{1/2}, 1)$. Let A_n be the collection of edges having non-empty intersection with the neighborhood

$$(3.9) \quad \mathcal{N}_{5d}\left(nS(0, v) \setminus \bigcup_{i=1}^\ell T_i\right)$$

and note that by construction, $|A_n| \leq c(d)mn^{d-2}$. We will soon take n to infinity, thus we lose no generality supposing n is large enough so that each $\text{d-cyl}(T_i, m + d^{1/2}, 1)$ is contained in $\text{d-cyl}(S(0, v), 1, n)$, so that in particular, each $E_i(\omega)$ is contained in the edge set of $\text{d-cyl}(S(0, v), 1, n)$ across all configurations ω .

One can repeat the argument at the end of the proof of Lemma 3.1 to show that the collection of edges $A_n \cup \left(\bigcup_{i=1}^\ell E_i(\omega)\right)$ which lie in $n\text{cyl}(S(0, v), 1)$ separate the vertex sets $\text{d-hemi}^\pm(S(0, v), 1, n)$ in $\text{d-cyl}(S(0, v), 1, n)$. Though we are dealing with many more boxes in this case, the complexity of the argument does not go up: we can always reduce to the case that our simple path last uses any vertex of $\text{d-hemi}^-(T_i, m + d^{1/2}, 1)$, and we may assume that our simple path must at some later point use a vertex within some $\text{d-hemi}(T_i, m + d^{1/2}, 1)$ which does not necessarily correspond to the same cube. Between these two points, we find that we must either use an edge in A_n , or an edge in one of the $E_i(\omega)$. Thus, we may conclude

$$(3.10) \quad \mathfrak{X}(0, v, n) \leq \sum_{i=1}^\ell \mathfrak{X}_{\text{hemi}}(T_i, m + d^{1/2}, 1) + c(d)mn^{d-2}$$

The chosen direction thus far has been arbitrary, so the preceding lemma applies to $\mathfrak{Y}(0, v, n)$ as well as to $\mathfrak{X}(0, v, n)$. We may take expectations of both sides and apply Lemma 3.1 to each term in the above sum, while also using our bound $\ell \leq (k+1)^{d-1}$

$$(3.11) \quad \mathbb{E}_p \mathfrak{X}(0, v, n) \leq \ell \mathbb{E}_p \mathfrak{Y}(0, v, m) + \ell c(d)m^{d-2} + c(d)mn^{d-2}$$

$$(3.12) \quad \leq (k+1)^{d-1} \mathbb{E}_p \mathfrak{Y}(0, v, m) + (k+1)^{d-1} c(d)m^{d-2} + c(d)mn^{d-2}$$

We divide through by n^{d-1} :

$$(3.13) \quad \frac{\mathbb{E}_p \mathfrak{X}(0, v, n)}{n^{d-1}} \leq (k+1)^{d-1} \frac{\mathbb{E}_p \mathfrak{Y}(0, v, m)}{n^{d-1}} + \frac{(k+1)^{d-1} m^{d-2} c(d)}{n^{d-1}} + \frac{c(d)m}{n}$$

$$(3.14) \quad \leq \left(\frac{k+1}{k}\right)^{d-1} k^{d-1} \cdot \frac{\mathbb{E}_p \mathfrak{Y}(0, v, m)}{n^{d-1}} + \left(\frac{k+1}{k}\right)^{d-1} \left(\frac{k}{n}\right)^{d-1} m^{d-2} c(d) + \frac{c(d)m}{n}$$

$$(3.15) \quad \leq \left(\frac{k+1}{k}\right)^{d-1} \frac{\mathbb{E}_p \mathfrak{Y}(0, v, m)}{m^{d-1}} + \left(\frac{k+1}{k}\right)^{d-1} \frac{c(d)}{m} + \frac{c(d)m}{n}$$

We first take the lim sup of both sides in n , and then the lim inf of both sides in m :

$$(3.16) \quad \limsup_{n \rightarrow \infty} \frac{\mathbb{E}_p \mathfrak{X}(0, v, n)}{n^{d-1}} \leq \frac{\mathbb{E}_p \mathfrak{Y}(0, v, m)}{m^{d-1}} + \frac{c(d)}{m}$$

$$(3.17) \quad \leq \liminf_{m \rightarrow \infty} \frac{\mathbb{E}_p \mathfrak{Y}(0, v, m)}{m^{d-1}}$$

the proof is complete upon dividing both sides by 2^{d-1} : setting $T(0, v) = S(0, v)$ gives us the existence of the limit in question, and interchanging $T(0, v)$ and $S(0, v)$ in the above argument tells us this limit does not depend on the chosen rotation. The finiteness of this limit can be seen in the following way: given a direction $v \in \mathbb{S}^{d-1}$, the collection of edges intersecting the neighborhood $\mathcal{N}_{5d}(nS(0, v))$ forms a cutset in $d\text{-cyl}(S(0, v), 1, n)$ separating $d\text{-hemi}^\pm(S(0, v), 1, n)$ and this cutset has cardinality bounded above by $c(d)n^{d-1}$ uniformly for all directions v . \square

We can immediately deduce that the function $\beta_{p,d}$ inherits the symmetries of \mathbb{Z}^d .

Corollary 3.3. *Let $d \geq 2$ and $p > p_c(d)$. For all $v \in \mathbb{S}^{d-1}$ and for all linear transformations $L : \mathbb{R}^d \rightarrow \mathbb{R}^d$ such that $L(\mathbb{Z}^d) = \mathbb{Z}^d$, we have $\beta_{p,d}(Lv) = \beta_{p,d}(v)$.*

Proof. Let $v \in \mathbb{S}^{d-1}$. Consider the chosen square $S(0, Lv)$ for the direction Lv . then $L^{-1}S(0, Lv)$ is a rotation of $S(0, v)$ contained in $\text{hyp}S(0, v)$. We write $T(0, v)$ in place of $L^{-1}S(0, Lv)$. From the preceding Proposition 3.2, we know $\mathbb{E}_p \mathfrak{X}_{\text{hemi}}(T(0, v), 1, n)/(2n)^{d-1}$ and $\mathbb{E}_p \mathfrak{X}(0, v, n)/(2n)^{d-1}$ have the same limit in n . Because L induces a graph automorphism of \mathbb{Z}^d , we have also that $\mathbb{E}_p \mathfrak{X}_{\text{hemi}}(T(0, v), 1, n) = \mathbb{E}_p \mathfrak{X}(0, Lv, n)$, so that

$$(3.18) \quad \lim_{n \rightarrow \infty} \frac{\mathbb{E}_p \mathfrak{X}(0, Lv, n)}{(2n)^{d-1}} = \lim_{n \rightarrow \infty} \frac{\mathbb{E}_p \mathfrak{X}(0, v, n)}{(2n)^{d-1}}$$

which gives the desired result. \square

3.3. The chosen rotation and properties of $\beta_{p,d}$. The function $\beta_{p,d}$ could have been defined using cylinders based at discs instead of squares, but this would have made it much harder to use the above subadditivity argument. Moreover, we will need similar arguments in Section 5. There is a tradeoff between the tidiness of these arguments and the somewhat artificial nature of the chosen rotation, and we feel we have taken the route which is ultimately cleanest.

Part of this tradeoff is that we need the chosen rotation $f : v \mapsto S(0, v)$ to vary over most of the sphere in a Lipschitz way. This allows us to show the sequence of functions $v \mapsto \mathbb{E}_p \mathfrak{X}(0, v, n)/(2n)^{d-1}$ on \mathbb{S}^{d-1} converges uniformly to $\beta_{p,d}$, which will in turn allow us to prove concentration estimates in Section 4. For topological reasons, it is not possible to have f vary in a Lipschitz way over the entire sphere, so we must first work over the closed upper hemisphere. Let \mathbb{H}_+^{d-1} be the closed upper hemisphere of \mathbb{S}^{d-1} , that is,

$$(3.19) \quad \mathbb{H}_+^{d-1} := \mathbb{S}^{d-1} \cap \{x \in \mathbb{R}^d : x_d \geq 0\}$$

A corollary of Proposition A.15 from the appendix is that we may define f over the upper hemisphere in a Lipschitz way. That is, if $v, w \in \mathbb{H}_+^{d-1}$, there exists a positive constant $M(d)$ so that if $|v - w|_2 < \epsilon$, we have

$$(3.20) \quad \partial S(0, v) \subset \mathcal{N}_{M\epsilon}(\partial S(0, w)) \quad \text{and} \quad \partial S(0, w) \subset \mathcal{N}_{M\epsilon}(\partial S(0, v))$$

where $\partial S(0, v)$ denotes the boundary of $S(0, v)$ within $\text{hyp}(S(0, v))$ and similarly for $\partial S(0, w)$. Now that the chosen rotation is defined over the closed upper hemisphere, we extend this definition to the rest of \mathbb{S}^{d-1} in a natural way. Let $A : \mathbb{R}^d \rightarrow \mathbb{R}^d$ be the linear transformation which is diagonal with respect to the coordinate basis, having all ones on the diagonal except the last entry which is negative one. Given $v \in \mathbb{S}^{d-1} \setminus \mathbb{H}_+^{d-1}$, we define $f(v) = S(0, v)$ by $AS(0, Av)$. With f now defined on all of \mathbb{S}^{d-1} , we note that the above Lipschitz property holds whenever v and w both lie in the closed upper hemisphere, or whenever v, w both lie in the open lower hemisphere.

Proposition 3.4. *For $d \geq 2$ and $p > p_c(d)$, the sequence of functions on \mathbb{S}^{d-1} defined by $v \mapsto \mathbb{E}_p \mathfrak{X}(0, v, n)/(2n)^{d-1}$ converge uniformly to $\beta_{p,d}$.*

Proof. Let $\epsilon > 0$, Let $v, w \in \mathbb{H}_+^{d-1}$ be such that $|v - w|_2 < \epsilon$. Fix a configuration ω and let $E_n(\omega)$ be minimal cutset which separates the hemispheres $d\text{-hemi}^\pm(S(0, v), 1, n)$ within $d\text{-cyl}(S(0, v), 1, n)$. Choose $E_n(\omega)$ so that $|E_n(\omega)|_\omega$ is $\mathfrak{X}(0, v, n)$ in the configuration ω . By our continuity hypothesis on the chosen rotation, we have that

$$(3.21) \quad n\text{cyl}(S(0, v), 1) \subset n(1 + M\epsilon)\text{cyl}(S(0, w), 1)$$

In particular, this implies $E_n(\omega)$ is contained within the edge set of $d\text{-cyl}(S(0, w), 1, n(1 + M\epsilon))$. As we have done before, we would like to use $E_n(\omega)$ in conjunction with a small collection of edges to produce a cut separating the hemispheres of $d\text{-cyl}(S(0, w), 1, n(1 + M\epsilon))$. We will actually use two other collections of edges to do this.

Let A_n be the collection of edges having non-empty intersection with

$$(3.22) \quad \mathcal{N}_{5d} [n(\partial\text{cyl}(S(0, v), 1)) \cap n(\text{slab}(S(0, v), M\epsilon))]$$

Here we suppose that ϵ is small enough so that $\text{slab}(S(0, v), M\epsilon)$ does not contain the top and bottom faces of the cube $\text{cyl}(S(0, v), 1)$. The neighborhood (3.22) is a slight thickening of an equatorial band of height $nM\epsilon$ in $n\partial\text{cyl}(S(0, v), 1)$. It follows that $|A_n| \leq c(d)M\epsilon n^{d-1}$.

Let B_n be the collection of edges having non-empty intersection with the neighborhood

$$(3.23) \quad \mathcal{N}_{5d} ([n(1 + M\epsilon)]S(0, w) \setminus n\text{cyl}(S(0, v), 1))$$

So that the above neighborhood (3.23) bridges the neighborhood (3.22) defining A_n and the equator of the larger cube $n(1 + M\epsilon)\text{cyl}(S(0, w), 1)$. By construction, we also have $|B_n| \leq c(d)M\epsilon n^{d-1}$. Figure 4 depicts the cut $E_n(\omega)$ with the edge sets A_n and B_n .

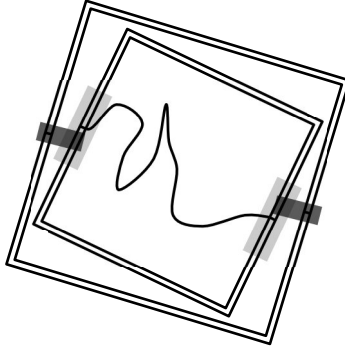


FIGURE 4. The cut $E_n(\omega)$ in the smaller cube is central. At the equator of the smaller cube, this cut meets with the edge set A_n , which is represented by the lightly shaded regions. The edge set A_n is joined to the equator of the larger cube by the edge set B_n , depicted as the darker shaded regions.

The edges of the union $E_n(\omega) \cup A_n \cup B_n$ contained in $d\text{-cyl}(S(0, w), 1, n(1 + M\epsilon))$ form a cutset separating the hemispheres $d\text{-hemi}^\pm(S(0, w), 1, n(1 + M\epsilon))$. The argument for this is nearly identical to the one given at the end of Lemma 3.1. Indeed, we are looking at nested cubes, the only difference being that one is tilted slightly relative to the other. This tilt necessitates using the edge set A_n in our case, with B_n playing the role of A_r from Lemma 3.1. With this established, we conclude

$$(3.24) \quad \mathfrak{X}(0, w, [n(1 + M\epsilon)]) \leq \mathfrak{X}(0, v, n) + c(d)M\epsilon n^{d-1}$$

so that by taking expectations, we have

$$(3.25) \quad \frac{\mathbb{E}_p \mathfrak{X}(0, w, [n(1 + M\epsilon)])}{(2[n(1 + M\epsilon)])^{d-1}} \leq \frac{\mathbb{E}_p \mathfrak{X}(0, v, n)}{(2n)^{d-1}} + c(d)M\epsilon$$

Taking $n \rightarrow \infty$, and using the symmetry in v and w , we have shown that when $v, w \in \mathbb{H}_+^{d-1}$ satisfy $|v - w|_2 < \epsilon$, we have $|\beta_{p,d}(v) - \beta_{p,d}(w)| < c(d)M\epsilon$. A symmetric argument shows that we have the same bounds whenever $v, w \in \mathbb{S}^{d-1} \setminus \mathbb{H}_+^{d-1}$ and $|v - w|_2 < \epsilon$.

Choose a finite collection of unit vectors $\{v_i\}_{i=1}^m$ (with $m = m(\epsilon)$), arranged so that if $v \in \mathbb{H}_+^{d-1}$, we can find $v_i \in \mathbb{H}_+^{d-1}$ so that $|v - v_i|_2 < \epsilon$, and if $v \in \mathbb{S}^{d-1} \setminus \mathbb{H}_+^{d-1}$, there is $v_i \in \mathbb{S}^{d-1} \setminus \mathbb{H}_+^{d-1}$ with $|v - v_i|_2 < \epsilon$. Take N large enough so that whenever $n \geq N$, we have for each i ,

$$(3.26) \quad \left| \frac{\mathbb{E}_p \mathfrak{X}(0, v_i, n)}{(2n)^{d-1}} - \beta_{p,d}(v_i) \right| < \epsilon$$

Let $v \in \mathbb{S}^{d-1}$ and take v_i in the same hemisphere so that $|v - v_i| < \epsilon$. Two applications of (3.25) show

$$(3.27) \quad \frac{\mathbb{E}_p \mathfrak{X}(0, v_i, \lceil n(1 + M\epsilon) \rceil (1 + M\epsilon))}{(2 \lceil n(1 + M\epsilon) \rceil (1 + M\epsilon))^{d-1}} - c(d)M\epsilon \leq \frac{\mathbb{E}_p \mathfrak{X}(0, v, \lceil n(1 + M\epsilon) \rceil)}{(2 \lceil n(1 + M\epsilon) \rceil)^{d-1}}$$

$$(3.28) \quad \leq \frac{\mathbb{E}_p \mathfrak{X}(0, v_i, n)}{(2n)^{d-1}} + c(d)M\epsilon$$

So that, by the hypotheses on the v_i and from the uniform continuity of β_p shown above on $\mathbb{S}^{d-1} \setminus \{s_d\}$, we have

$$(3.29) \quad \beta_p(v) - \epsilon - 2c(d)M\epsilon \leq \frac{\mathbb{E}_p \mathfrak{X}(0, v, \lceil n(1 + M\epsilon) \rceil)}{(2 \lceil n(1 + M\epsilon) \rceil)^{d-1}} \leq \beta_p(v) + \epsilon + 2c(d)M\epsilon$$

which establishes the desired uniform convergence. \square

We now extend $\beta_{p,d}$ to a function on all of \mathbb{R}^d via homogeneity:

$$(3.30) \quad \beta_{p,d}(x) := \begin{cases} |x|_2 \beta_{p,d}(x/|x|_2) & |x|_2 > 0 \\ 0 & |x|_2 = 0 \end{cases}$$

where $x \in \mathbb{R}^d$. To show $\beta_{p,d}$ defines a norm on \mathbb{R}^d , we show $\beta_{p,d}$ satisfies the so-called weak triangle inequality. For two distinct points $a, b \in \mathbb{R}^d$, we let $[ab]$ denote the closed line segment in \mathbb{R}^d connecting a and b , and for $a, b, c \in \mathbb{R}^d$ not co-linear, we let $[abc]$ denote the closed triangle in \mathbb{R}^d with vertices a, b, c . For a triangle $[abc]$, we define v_a to be the outward pointing unit normal to the side $[bc]$ within the plane spanned by $[abc]$. We define v_b and v_c analogously.

Proposition 3.5. *For $d \geq 2$ and $p > p_c(d)$, the function $\beta_{p,d} : \mathbb{S}^{d-1} \rightarrow [0, \infty)$ satisfies the weak triangle inequality. That is, for $a, b, c \in \mathbb{R}^d$ not co-linear,*

$$\mathcal{H}^1([bc])\beta_{p,d}(v_a) \leq \mathcal{H}^1([ac])\beta_{p,d}(v_b) + \mathcal{H}^1([ab])\beta_{p,d}(v_c)$$

Proof. The proof here is identical to Proposition 11.6 in [18], see also Proposition 4.5 of [56]. \square

The following is a consequence of the weak triangle inequality.

Proposition 3.6. *For $d \geq 2$ and $p > p_c(d)$, the function $\beta_{p,d} : \mathbb{R}^d \rightarrow [0, \infty]$ defines a norm on \mathbb{R}^d .*

Proof. We combine Proposition 3.5 with Corollary 11.7 of [18] to conclude that $\beta_{p,d}$ is a convex function on \mathbb{R}^d . To show non-degeneracy of $\beta_{p,d}$ then, it suffices to show non-degeneracy in the cardinal directions. Thanks to the symmetries of our norm (Corollary 3.3) it suffices to show non-degeneracy in a single cardinal direction. But this follows from Theorem 7.68 of [37] in conjunction with Menger's theorem. \square

That $\beta_{p,d}$ is a norm allows us to define the associated surface energy $\mathcal{I}_{p,d}$, as in Section 2. This gives rise to the unit Wulff crystal $\widehat{W}_{p,d}$ corresponding to $\beta_{p,d}$, and we define the *Wulff crystal* $W_{p,d}$ to be the dilate of $\widehat{W}_{p,d}$ about the origin so that $\mathcal{L}^d(W_{p,d}) = 2^d/d!$. The Wulff crystal $W_{p,d}$ is the limit shape which shows up in Theorems 1.2 and 2.2. So that these theorems make sense, we must know that $W_{p,d}$ is contained in $[-1, 1]^d$.

Lemma 3.7. *For $d \geq 2$ and $p > p_c(d)$, the Wulff crystal $W_{p,d}$ is contained in $[-1, 1]^d$.*

Proof. From Corollary 3.3 we use the symmetries of $\beta_{p,d}$ to deduce that the unit volume Wulff crystal $\widehat{W}_{p,d}$ satisfies

$$\beta_{p,d}(e_1)B_1 \subset \widehat{W}_{p,d} \subset \beta_{p,d}(e_1)B_\infty$$

where B_1 and B_∞ respectively denote unit ℓ^1 - and unit ℓ^∞ -balls in \mathbb{R}^d centered at the origin, and where $e_1 = (1, 0, \dots, 0) \in \mathbb{R}^d$. The claim follows from the fact that $\mathcal{L}^d(B_1) = 2^d/d!$. \square

We may use $\mathcal{I}_{p,d}$ to define an analogous notion of conductance in the continuum: for $E \subset \mathbb{R}^d$ a set of finite perimeter, we define the *conductance* of E as $\mathcal{I}_{p,d}(E)/\theta_p(d)\mathcal{L}^d(E)$. As stated in the introduction, proving Theorem 1.2 amounts to showing a strong relationship between our notions of conductance in the discrete and continuum realms.

4. CONCENTRATION ESTIMATES FOR $\beta_{p,d}$

In the present section and the next, we relate our new notion of conductance in the continuum back to the notion in the discrete setting. We must first derive concentration estimates for the random variables used to define $\beta_{p,d}$. We follow an argument of Zhang in Section 9 of [67], using other results from his paper in conjunction with the following concentration inequality due to Talagrand. We find it aesthetically pleasing to use a result powered by isoperimetry towards another isoperimetric result in a much different setting.

Theorem 4.1. *Let (V, E) be a graph with $\{X_e\}_{e \in E}$ a collection of iid Bernoulli(p) random variables. Let \mathcal{S} denote a family of sets of edges and for $S \in \mathcal{S}$, let $X_S = \sum_{e \in S} X_e$. Let $Z_S = \inf_{S \in \mathcal{S}} X_S$, and let M be a median of Z_S . There is a constant $c = c(p) > 0$ so that for all $u > 0$,*

$$\mathbb{P}_p(|Z_S - M| \geq u) \leq 4 \exp\left(-c \min\left(\frac{u^2}{\alpha}, u\right)\right)$$

where $\alpha = \sup_{S \in \mathcal{S}} |S|$.

This theorem was stated in the more general context of first passage percolation in Section 8.3 of [58]; we have only presented the version for Bernoulli percolation. It is clear how our random variables $\mathfrak{X}_{\text{hemi}}$ and $\mathfrak{X}_{\text{face}}$ could be expressed as Z_S for some family of edge sets \mathcal{S} . However, use of Theorem 4.1 requires control over the size of the largest edge set in \mathcal{S} through the term α . In our case, we must control the size of the largest minimal cut separating two hemispheres (or opposing faces) of a cube.

We treat $v \in \mathbb{S}^{d-1}$ as fixed for now, and we only consider cubes centered at the origin. To simplify our notation further, we write $\mathfrak{X}(0, v, n)$ as \mathfrak{X}_n . Following Zhang in [67], we'll first use the above theorem to prove concentration for a modification of the sequence \mathfrak{X}_n in which we examine only cutsets using a surface order number of edges. Let $\gamma > 0$, and let $\mathcal{S}_n(\gamma)$ be the family of cutsets in $d\text{-cyl}(S(0, v), 1, n)$ which separate the hemispheres $d\text{-hemi}^\pm(S(0, v), 1, n)$, and which satisfy $|S| \leq \gamma(2n)^{d-1}$. Define

$$(4.1) \quad Z_n^{(\gamma)}(\omega) := \inf_{S \in \mathcal{S}_n(\gamma)} |S|_\omega$$

We can apply Theorem 4.1 to each $Z_n^{(\gamma)}$, noting that $\alpha \leq \gamma(2n)^{d-1}$ by construction.

Proposition 4.2. *Let $\epsilon, \gamma > 0$. There are positive constants $c_1(p, \gamma, \epsilon)$ and $c_2(p, \gamma, \epsilon)$ so that*

$$\mathbb{P}_p\left(\frac{|Z_n^{(\gamma)} - \mathbb{E}_p Z_n^{(\gamma)}|}{(2n)^{d-1}} \geq \epsilon\right) \leq c_1 \exp\left(-c_2 n^{(d-1)/3}\right)$$

Proof. We follow the argument at the beginning of Section 9 in [67]. Let us write $A = A(n) = (2n)^{d-1}$ for the \mathcal{H}^{d-1} -measure (“area”) of the dilated square $nS(0, v)$. Let $M_n^{(\gamma)}$ be a median for $Z_n^{(\gamma)}$. Then,

$$(4.2) \quad |\mathbb{E}_p Z_n^{(\gamma)} - M_n^{(\gamma)}| \leq \mathbb{E}_p |Z_n^{(\gamma)} - M_n^{(\gamma)}|$$

$$(4.3) \quad \leq \sum_{j=1}^{\lfloor A^{2/3} \rfloor} \mathbb{P}_p(|Z_n^{(\gamma)} - M_n^{(\gamma)}| \geq j) + \sum_{j=\lfloor A^{2/3} \rfloor}^{\infty} \mathbb{P}_p(|Z_n^{(\gamma)} - M_n^{(\gamma)}| \geq j)$$

Apply Theorem 4.1 to the right-most sum above:

$$(4.4) \quad |\mathbb{E}_p Z_n^{(\gamma)} - M_n^{(\gamma)}| \leq A^{2/3} + 4 \sum_{j=\lfloor A^{4/3} \rfloor}^{\infty} \exp\left(-c \frac{j}{\gamma A}\right) + 4 \sum_{j=\lfloor A^{2/3} \rfloor}^{\infty} \exp(-cj)$$

$$(4.5) \quad \leq A^{2/3} + \frac{4}{1 - \exp(-c/\gamma A)} \exp(-cA^{1/3}/\gamma) + \frac{4}{1 - \exp(-c)} \exp(-cA^{2/3})$$

Thus for all n sufficiently large, in a way which depends on p and γ , we have $|\mathbb{E}_p Z_n^{(\gamma)} - M_n^{(\gamma)}| \leq (3/2)A^{2/3}$. When n is large enough, by the triangle inequality and the above computation,

$$(4.6) \quad \mathbb{P}_p(|Z_n^{(\gamma)} - \mathbb{E}_p Z_n^{(\gamma)}| \geq 4A^{2/3}) \leq \mathbb{P}_p(|Z_n^{(\gamma)} - M_n^{(\gamma)}| + |M_n^{(\gamma)} - \mathbb{E}_p Z_n^{(\gamma)}| \geq 4A^{2/3})$$

$$(4.7) \quad \leq \mathbb{P}_p(|Z_n^{(\gamma)} - M_n^{(\gamma)}| \geq 2A^{2/3})$$

Another application of Theorem 4.1 gives that, when n is sufficiently large depending on p and γ ,

$$(4.8) \quad \mathbb{P}_p(|Z_n^{(\gamma)} - \mathbb{E}_p Z_n^{(\gamma)}| \geq 4A^{2/3}) \leq 4 \exp\left(-c \min\left(\frac{4}{\gamma}A^{1/3}, 2A^{2/3}\right)\right)$$

and we complete the proof dividing by A within the probability on the left. \square

In order to use Proposition 4.2 to obtain a statement about the \mathfrak{X}_n , we need Proposition 4.2 in the paper of Rossignol and Thérét [56], which we specialize to Bernoulli percolation. As these authors say, this proposition is a consequence of Proposition 5.8 of Kesten’s [41] in dimension two, with the main difficulties in higher dimensions settled through Theorem 1 of Zhang’s [67]. This latter argument due to Zhang involves a construction lying at the heart of Section 6, and which ultimately allows us to pass from Cheeger optimizers to continuous objects having some regularity.

For a percolation configuration ω , let $N_n(\omega)$ denote the minimum cardinality $|S|$ over all cutsets S in $d\text{-cyl}(S(0, v), 1, n)$ separating $d\text{-hemi}^\pm(S(0, v), 1, n)$ such that $|S|_\omega = \mathfrak{X}_n(\omega)$. The following is Proposition 4.2 from [56].

Proposition 4.3. *Let $d \geq 2$ and let $p > p_c(d)$. There are positive constants $\gamma(p, d)$, $c_1(p, d)$ and $c_2(p, d)$ so that for all $u > 0$,*

$$\mathbb{P}_p(N_n \geq \gamma u \text{ and } \mathfrak{X}_n \leq u) \leq c_1 \exp(-c_2 u)$$

Using this with Proposition 4.2, we deduce the following.

Corollary 4.4. *Let $d \geq 2$, $p > p_c(d)$, $v \in \mathbb{S}^{d-1}$ and let $\epsilon > 0$. There are positive constants $c_1(p, d, \epsilon)$ and $c_2(p, d, \epsilon)$ so that*

$$\mathbb{P}_p\left(\frac{|\mathfrak{X}(0, v, n) - \mathbb{E}_p \mathfrak{X}(0, v, n)|}{(2n)^{d-1}} \geq \epsilon\right) \leq c_1 \exp(-c_2 n^{(d-1)/3})$$

Proof. We fix $v \in \mathbb{S}^{d-1}$ for now and abbreviate $\mathfrak{X}(0, v, n)$ as \mathfrak{X}_n . Note that uniformly in $v \in \mathbb{S}^{d-1}$, there exists a constant $c(d)$ so that $\mathfrak{X}_n(\omega) \leq c(d)n^{d-1}$ for all configurations ω . To see this, take all edges which have non-empty intersection with a $5d$ -neighborhood of the dilated square $nS(0, v)$. These edges form a cutset

between the hemispheres of $d\text{-cyl}(S(0, v), 1, n)$ of size at most $c(d)n^{d-1}$, with this constant not depending on the direction v .

Let $u = c(d)n^{d-1}$ and apply Proposition 4.3 to obtain γ depending on p and d such that

$$(4.9) \quad \mathbb{P}_p(N_n \geq \gamma c(d)n^{d-1}) \leq c_1 \exp(-c_2 c(d)n^{d-1})$$

We'll use this bound shortly. For this γ and for $\epsilon > 0$, use Proposition 4.2 to obtain positive constants $c_1(p, \gamma, \epsilon)$ and $c_2(p, \gamma, \epsilon)$ so that

$$(4.10) \quad \mathbb{P}_p\left(\frac{|\mathfrak{X}_n - \mathbb{E}_p \mathfrak{X}_n|}{(2n)^{d-1}} \geq \epsilon\right) \leq \mathbb{P}_p(Z_n^{(\gamma)} \neq \mathfrak{X}_n) + \mathbb{P}_p\left(\frac{|Z_n^{(\gamma)} - \mathbb{E}_p Z_n^{(\gamma)}|}{(2n)^{d-1}} \geq \epsilon\right)$$

$$(4.11) \quad \leq \mathbb{P}_p(Z_n^{(\gamma)} \neq \mathfrak{X}_n) + c_1 \exp(-c_2 n^{(d-1)/3})$$

Observe that $\{Z_n \neq \mathfrak{X}_n\} \subset \{N_n \geq \gamma c(d)n^{d-1}\}$, so that

$$(4.12) \quad \mathbb{P}_p\left(\frac{|\mathfrak{X}_n - \mathbb{E}_p \mathfrak{X}_n|}{(2n)^{d-1}} \geq \epsilon\right) \leq \mathbb{P}_p(N_n \geq \gamma c(d)n^{d-1}) + c_1 \exp(-c_2 n^{(d-1)/3})$$

$$(4.13) \quad \leq c_1 \exp(-c_2 c(d)n^{d-1}) + c_1 \exp(-c_2 n^{(d-1)/3})$$

The proof is complete. \square

We obtain the desired concentration estimates by combining Corollary 4.4 with the fact that the functions $v \mapsto \mathbb{E}_p \mathfrak{X}(0, v, n)/(2n)^{d-1}$ converge uniformly to $\beta_{p,d}$.

Theorem 4.5. *Let $d \geq 2$, $p > p_c(d)$, $v \in \mathbb{S}^{d-1}$ and let $\epsilon > 0$. There are positive constants $c_1(p, d, \epsilon)$, $c_2(p, d, \epsilon)$ so that*

$$\mathbb{P}_p\left(\left|\frac{\mathfrak{X}(0, v, n)}{(2n)^{d-1}} - \beta_{p,d}(v)\right| \geq \epsilon\right) \leq c_1 \exp(-c_2 n^{(d-1)/3})$$

Proof. By Proposition 3.4, we may choose N large depending on ϵ so that for all $v \in \mathbb{S}^{d-1}$, $n \geq N$ implies

$$(4.14) \quad \left|\frac{\mathbb{E}_p \mathfrak{X}(0, v, n)}{(2n)^{d-1}} - \beta_{p,d}(v)\right| < \epsilon/2$$

For $n \geq N$ we have

$$(4.15) \quad \mathbb{P}_p\left(\left|\frac{\mathfrak{X}(0, v, n)}{(2n)^{d-1}} - \beta_{p,d}(v)\right| \geq \epsilon\right) \leq \mathbb{P}_p\left(\frac{|\mathfrak{X}(0, v, n) - \mathbb{E}_p \mathfrak{X}(0, v, n)|}{(2n)^{d-1}} + \left|\frac{\mathbb{E}_p \mathfrak{X}(0, v, n)}{(2n)^{d-1}} - \beta_{p,d}(v)\right| \geq \epsilon\right)$$

$$(4.16) \quad \leq \mathbb{P}_p\left(\frac{|\mathfrak{X}(0, v, n) - \mathbb{E}_p \mathfrak{X}(0, v, n)|}{(2n)^{d-1}} \geq \epsilon/2\right)$$

The proof is complete applying the concentration estimates Corollary 4.4 to the right-hand side. \square

It will be useful to generalize Theorem 4.5 slightly to boxes whose side-length is not necessarily an integer, and whose centers do not necessarily lie in \mathbb{Z}^d . The following is the main result of this section.

Theorem 4.6. *Let $d \geq 2$ and $p > p_c(d)$. For $x \in \mathbb{R}^d$, $v \in \mathbb{S}^{d-1}$ and $\epsilon > 0$, there are positive constants $c_1(p, d, \epsilon)$, $c_2(p, d, \epsilon)$ so that*

$$\mathbb{P}_p\left(\left|\frac{\mathfrak{X}(x, v, r)}{(2r)^{d-1}} - \beta_{p,d}(v)\right| \geq \epsilon\right) \leq c_1 \exp(-c_2 r^{(d-1)/3})$$

Proof. Fix $v \in \mathbb{S}^{d-1}$; we claim that for any $r > 0$ and $x \in \mathbb{R}^d$, there is a constant $c(d)$ so that

$$(4.17) \quad \mathfrak{X}(x, v, \lceil r \rceil) - c(d)r^{d-2} \leq \mathfrak{X}(x, v, r) \leq \mathfrak{X}(x, v, \lfloor r \rfloor) + c(d)r^{d-2}$$

To see this, let A_r be the collection of edges having non-empty intersection with

$$(4.18) \quad \mathcal{N}_{5d}(rS(0, v) \setminus [r]S(0, v))$$

and let $E_{[r]}(\omega)$ be a cutset in $d\text{-cyl}(S(0, v), 1, [r])$ separating $d\text{-hemi}^\pm(S(0, v), 1, [r])$, and such that $|E_{[r]}(\omega)|_\omega = \mathfrak{X}(x, v, [r])$ in the configuration ω .

Our standard argument from Lemma 3.1 tells us that the edges of $E_{[r]}(\omega) \cup A$ contained in $d\text{-cyl}(S(0, v), 1, r)$ separate the hemispheres $d\text{-hemi}^\pm(S(0, v), 1, r)$. That $|A_r| \leq c(d)r^{d-2}$ establishes the claimed upper bound on $\mathfrak{X}(0, v, r)$, and we obtain the lower bound through a similar procedure.

We may also use the argument of Lemma 3.1 to observe that for each $x \in \mathbb{R}^d$ and $r > 0$, there exists $x' \in \mathbb{Z}^d$ so that

$$(4.19) \quad \mathfrak{X}(x', v, r + d^{1/2}) - c(d)r^{d-2} \leq \mathfrak{X}(x, v, r) \leq \mathfrak{X}(x', v, r - d^{1/2}) + c(d)r^{d-2}$$

Apply (4.17), to conclude

$$(4.20) \quad \mathfrak{X}(x', v, \lceil r + d^{1/2} \rceil) - c(d)r^{d-2} \leq \mathfrak{X}(x, v, r) \leq \mathfrak{X}(x', v, \lfloor r - d^{1/2} \rfloor) + c(d)r^{d-2}$$

As $x' \in \mathbb{Z}^d$, the random variables $\mathfrak{X}(x', v, \lceil r + d^{1/2} \rceil)$ and $\mathfrak{X}(x', v, \lfloor r - d^{1/2} \rfloor)$ have the same law as $\mathfrak{X}(0, v, \lceil r + d^{1/2} \rceil)$ and $\mathfrak{X}(0, v, \lfloor r - d^{1/2} \rfloor)$ respectively, so the estimates of Theorem 4.5 apply here. Within the high probability event

$$(4.21) \quad \left\{ \left| \frac{\mathfrak{X}(x', v, \lceil r + d^{1/2} \rceil)}{(2\lceil r + d^{1/2} \rceil)^{d-1}} - \beta_{p,d}(v) \right| < \epsilon \right\} \cap \left\{ \left| \frac{\mathfrak{X}(x', v, \lfloor r - d^{1/2} \rfloor)}{(2\lfloor r - d^{1/2} \rfloor)^{d-1}} - \beta_{p,d}(v) \right| < \epsilon \right\}$$

and taking r sufficiently large in a way depending on ϵ and d , we obtain

$$(4.22) \quad \beta_{p,d}(v) - 3\epsilon \leq \frac{\mathfrak{X}(x, v, r)}{(2r)^{d-1}} \leq \beta_{p,d}(v) + 3\epsilon$$

which completes the proof. \square

5. CONSEQUENCES OF CONCENTRATION ESTIMATES

We now derive important consequences of Theorem 4.6. In Section 5.1, we will obtain information about the random variables $\mathfrak{X}_{\text{hemi}}$ and $\mathfrak{X}_{\text{face}}$ for cylinders of small height over squares and discs. Such results will be important for arguments in Section 8 in which these cylinders are carefully placed on the boundary of a set of finite perimeter.

In Section 5.2, we use a result powered by Theorem 4.6 in conjunction with Gandolfi's results (presented in the appendix) on the density of \mathbf{C}_∞ to obtain high probability upper bounds on the Cheeger constant. Specifically, we will see that any polytope $P \subset [-1, 1]^d$ satisfying $\mathcal{L}^d(P) \leq 2^d/d!$ gives rise to such a bound. Specializing these results to a sequence of polytopes which are progressively better approximates of the Wulff crystal, we obtain what will later be shown to be a tight upper bound on the asymptotic value of the Cheeger constant.

5.1. Lower bounds for cuts in thin cylinders. We first observe how our concentration estimates apply to cylinders over squares of small height for random variables of the form $\mathfrak{X}_{\text{face}}$.

Lemma 5.1. *Let $d \geq 2$, $p > p_c(d)$ and let $\epsilon > 0$. There exists $\eta(p, d, \epsilon) > 0$ small and positive constants $c_1(p, d, \epsilon)$, $c_2(p, d, \epsilon)$ so that for all $x \in \mathbb{R}^d$ and all $v \in \mathbb{S}^{d-1}$ and $h \leq \eta$, we have*

$$\mathbb{P}_p \left(\mathfrak{X}_{\text{face}}(S(x, v), h, r) \leq (1 - \epsilon) \mathcal{H}^{d-1}(rS(x, v)) \beta_{p,d}(v) \right) \leq c_1 \exp \left(-c_2 r^{(d-1)/3} \right)$$

Proof. Write S for $S(x, v)$, fix a configuration ω and consider a minimal cutset $E_r(\omega)$ in $\text{d-cyl}(S, h, r)$ which separates the faces $\text{d-face}^\pm(S, h, r)$ and such that $|E_r(\omega)|_\omega = \mathfrak{X}_{\text{face}}(S, h, r)$ in the configuration ω .

Recall that S_h^+ and S_h^- are the top and bottom faces of the cylinder $\text{cyl}(S, h)$. We remove these faces from $\partial\text{cyl}(S, h)$ and consider the collection of edges A having non-empty intersection with the neighborhood

$$(5.1) \quad \mathcal{N}_{5d}(r(\partial\text{cyl}(S, h) \setminus (S_h^+ \cup S_h^-)))$$

The edges of $E_r(\omega) \cup A$ contained in $\text{d-cyl}(S, h, r)$ separate the hemispheres $\text{d-hemi}^\pm(S, h, r)$ within $\text{d-cyl}(S, h, r)$. It follows that these edges also separate the hemispheres $\text{d-hemi}^\pm(S, 1, r)$ in the larger cylinder $\text{d-cyl}(S, 1, r)$. By construction, the cardinality of A is at most $c(d)hr^{d-1}$, so that

$$(5.2) \quad \mathfrak{X}(x, v, r) \leq \mathfrak{X}_{\text{face}}(S, h, r) + c(d)hr^{d-1}$$

Thus,

$$(5.3) \quad \{\mathfrak{X}_{\text{face}}(S, h, r) \leq (1 - \epsilon)\mathcal{H}^{d-1}(rS)\beta_{p,d}(v)\} \subset \{\mathfrak{X}(x, v, r) \leq (1 - \epsilon)\mathcal{H}^{d-1}(rS)\beta_{p,d}(v) + c(d)hr^{d-1}\}$$

$$(5.4) \quad \subset \{\mathfrak{X}(x, v, r) \leq (1 - \epsilon/2)\mathcal{H}^{d-1}(rS)\beta_{p,d}(v)\}$$

where we have chosen h small depending on p, d, ϵ to obtain the second line. We complete the proof by applying Theorem 4.6 to the event on the second line. \square

We now prove an analogue of Lemma 5.1 for cylinders of small height whose bases are discs instead of squares. In the following proposition, $D(x, v)$ shall be the isometric image of a $(d - 1)$ -ball of radius one, translated so that its center is at x and rotated so that v is normal to $\text{hyp}(D(x, v, \rho))$.

Proposition 5.2. *Let $d \geq 2$, $p > p_c(d)$ and let $\epsilon > 0$. Consider the disc $D(x, v)$ of radius one for $x \in \mathbb{R}^d$ and $v \in \mathbb{S}^{d-1}$. There exists $\eta(p, d, \epsilon) > 0$ small and positive constants $c_1(p, d, \epsilon)$ and $c_2(p, d, \epsilon)$ so that $h \leq \eta$ implies*

$$\mathbb{P}_p(\mathfrak{X}_{\text{face}}(D(x, v), h, r) \leq (1 - \epsilon)\mathcal{H}^{d-1}(rD(x, v))\beta_{p,d}(v)) \leq c_1 \exp(-c_2 r^{(d-1)/3})$$

Proof. Let $\epsilon' > 0$, write $D = D(x, v)$ and let D' denote the closed disc in \mathbb{R}^{d-1} defined by $\{x \in \mathbb{R}^{d-1} : |x| \leq 1\}$. Let $\varphi : D' \rightarrow \mathbb{R}^d$ be an isometry such that $\varphi(D') = D$. Let Δ^k denote the collection of closed dyadic squares in \mathbb{R}^{d-1} at scale k , so that each square is a translate of $2^{-k}[-1, 1]^{d-1}$. We choose $k \in \mathbb{N}$ large enough (depending on ϵ' and d) so that

$$(5.5) \quad \mathcal{L}^{d-1}\left(D' \setminus \bigcup_{S' \in \Delta^k, S' \subset D'} S'\right) \leq \epsilon' \mathcal{L}^{d-1}(D')$$

Let us enumerate the squares $S' \in \Delta^k$ with $S' \subset D'$ as S'_1, \dots, S'_m , noting that the number m of these squares depends on ϵ' and d . We shrink each cube slightly to form a new disjoint collection $\{S''_i\}_{i=1}^m$ of closed squares. Specifically, S''_i shall be the $(1 - \delta)$ -dilate of S'_i about its center.

For each i , let $S_i = \varphi(S''_i)$, so that for δ chosen sufficiently small (also depending on ϵ' and d),

$$(5.6) \quad \mathcal{H}^{d-1}\left(D \setminus \bigcup_{i=1}^m S_i\right) \leq 2\epsilon' \mathcal{H}^{d-1}(D)$$

Let $\alpha = (1 - \delta)2^{-k}$. We can arrange that the isometry φ is compatible with the chosen rotation, in the sense that for each i , we have $S_i = \alpha S(y_i, v)$.

Let $\epsilon > 0$, choose $\eta = \eta(p, d, \epsilon/2)$ as in Lemma 5.1 and let $h \leq \alpha\eta$. For a fixed configuration ω , let $E_r(\omega)$ be a minimal cutset separating the faces $\text{d-face}^\pm(D, h, r)$ within $\text{d-cyl}(D, h, r)$, so that $|E^{(r)}(\omega)|_\omega = \mathfrak{X}_{\text{face}}(D, h, r)$ in the configuration ω . Let $E_r^{(i)}(\omega)$ denote the set of all edges of $E_r(\omega)$ which are contained in

the edge set $\text{d-cyl}(S_i, h, r)$. Each $E_r^{(i)}$ separates the faces of $\text{d-face}^\pm(S_i, h, r)$ within $\text{d-cyl}(S_i, h, r)$, so from the disjointness of the collection $\{S_i\}_{i=1}^m$, we have

$$(5.7) \quad \sum_{i=1}^m \mathfrak{X}_{\text{face}}(S_i, h, r) \leq \mathfrak{X}_{\text{face}}(D, h, r)$$

Thus,

$$(5.8) \quad \mathbb{P}_p(\mathfrak{X}_{\text{face}}(D, h, r) \leq (1 - \epsilon)\mathcal{H}^{d-1}(rD)\beta_{p,d}(v))$$

$$(5.9) \quad \leq \mathbb{P}_p\left(\sum_{i=1}^m \mathfrak{X}_{\text{face}}(S_i, h, r) \leq (1 - \epsilon)\mathcal{H}^{d-1}(rD)\beta_{p,d}(v)\right)$$

$$(5.10) \quad \leq \mathbb{P}_p\left(\sum_{i=1}^m \mathfrak{X}_{\text{face}}(S_i, h, r) \leq \frac{(1 - \epsilon)}{1 - 2\epsilon'} \sum_{i=1}^m \mathcal{H}^{d-1}(rS_i)\beta_{p,d}(v)\right)$$

with the last line following from our choice of δ and the squares S'_i . As $\mathcal{H}^{d-1}(rS_i)$ is the same for all i , we use a union bound to obtain

$$(5.11) \quad \mathbb{P}_p(\mathfrak{X}_{\text{face}}(D, h, r) \leq (1 - \epsilon)\mathcal{H}^{d-1}(rD)\beta_{p,d}(v))$$

$$(5.12) \quad \leq \sum_{i=1}^m \mathbb{P}_p\left(\mathfrak{X}_{\text{face}}(S_i, h, r) \leq \left(\frac{1 - \epsilon}{1 - 2\epsilon'}\right)\beta_{p,d}(v)\mathcal{H}^{d-1}(rS_i)\right)$$

$$(5.13) \quad \leq \sum_{i=1}^m \mathbb{P}_p\left(\mathfrak{X}_{\text{face}}(S_i, h, r) \leq (1 - \epsilon/2)\beta_{p,d}(v)\mathcal{H}^{d-1}(rS_i)\right)$$

To obtain the second line, we have taken ϵ' small enough so that $1 - \epsilon/2 > \frac{1-\epsilon}{1-2\epsilon'}$. Thus, m and α now depend on ϵ and d . We use the fact that $S_i = \alpha S(y_i, v)$ for each i :

$$(5.14) \quad \mathbb{P}_p(\mathfrak{X}_{\text{face}}(D, h, r) \leq (1 - \epsilon)\mathcal{H}^{d-1}(rD)\beta_{p,d}(v))$$

$$(5.15) \quad \leq \sum_{i=1}^m \mathbb{P}_p\left(\mathfrak{X}_{\text{face}}(S(y_i, v), h/\alpha, \alpha r) \leq (1 - \epsilon/2)\beta_{p,d}(v)\mathcal{H}^{d-1}(\alpha r S(y_i, v))\right)$$

We have chosen h so that $h/\alpha \leq \eta$, and thus we may apply Lemma 5.1 to each summand on the left, using αr in place of r and $\epsilon/2$ in place of ϵ :

$$(5.16) \quad \mathbb{P}_p(\mathfrak{X}_{\text{face}}(D, h, r) \leq (1 - \epsilon)\mathcal{H}^{d-1}(rD)\beta(v)) \leq mc_1 \exp(-c_2(\alpha r)^{(d-1)/3})$$

The proof is complete upon renaming these constants and making note of their dependencies. We take $\alpha\eta$ to be the η in the statement of the proposition. \square

Until this point, we have only used half of our concentration estimates from Section 4, in the sense that we have only used these estimates to show that with high probability, the random variables \mathfrak{X} cannot be too small. In the next subsection, we put the complementary estimates to good use.

5.2. Upper bounds on $\widehat{\Phi}_n$, or efficient carvings of ice. We define a *convex polytope* to be a compact subset of \mathbb{R}^d which may be written as a finite intersection of closed half-spaces. We define a *polytope* to be a compact subset of \mathbb{R}^d which may be written as finite union of convex polytopes. In particular, we do not require polytopes to be connected subsets of \mathbb{R}^d , but we say that a polytope is *connected* if its interior is a connected subset of \mathbb{R}^d . We say that $P \subset \mathbb{R}^d$ is a *d-polytope* if it is non-degenerate ($\mathcal{L}^d(P) > 0$).

As stated at the beginning of the section, our goal is now to use a sufficiently small polytope $P \subset [-1, 1]^d$ to obtain upper bounds on $\widehat{\Phi}_n$. We will use P to obtain a valid subgraph of \mathbf{C}_n , and we must control both the open edge boundary of this subgraph as well as its volume. Equivalently, we view \mathbf{C}_n as a block of ice, and we use the dilate nP as a blueprint for carving this block.

Our first aim is to perform an “efficient” carving at the boundary of nP , and this is where the other side of our concentration estimates are used. It is convenient to prove a result, Proposition 5.4, which allows us to work on each face of the polytope P individually. First, we will need lemma to settle a slightly technical detail.

For a d -polytope P in \mathbb{R}^d , we say that $\eta > 0$ is *good for P* if each point in the closed neighborhood $\mathcal{N}_\eta(\partial P)$ possesses a unique closest point in ∂P . Likewise, suppose σ is a $(d-1)$ -polytope in \mathbb{R}^d , i.e. σ is the isometric image of a $(d-1)$ -polytope in \mathbb{R}^{d-1} . If $\partial\sigma$ denotes the boundary of σ within $\text{hyp}(\sigma)$, we say $\eta > 0$ is *good for σ* if every point in the closed neighborhood $\mathcal{N}_\eta(\partial\sigma)$ possesses a unique closest point in $\partial\sigma$. It is easy to see that if η is good for P or σ , so too is any smaller η' . The following lemma is essentially a version of the ϵ -neighborhood theorem of differential topology for polytopes.

Lemma 5.3. *For any connected d -polytope P , there is an $\eta(P) > 0$ so that η is good for P . Likewise, for any connected $(d-1)$ -polytope σ , there is an $\eta(\sigma) > 0$ so that η is good for σ .*

Proof. This follows from the proof of Proposition 4.8 in [23]. \square

We’ll think of a $(d-1)$ -polytope $\sigma \subset \mathbb{R}^d$ as one of the faces of a d -polytope P . We let v_σ denote one of the unit vectors orthogonal to $\text{hyp}(\sigma)$. Thanks to the symmetries of $\beta_{p,d}$, our choice of unit vector is immaterial to the following proposition.

Proposition 5.4. *Let $d \geq 2$ and let $p > p_c(d)$. Let $\sigma \subset \mathbb{R}^d$ be a connected $(d-1)$ -polytope, and let $\epsilon > 0$. There is a positive constant $\eta(p, d, \epsilon, \sigma)$ and another connected $(d-1)$ -polytope $\tilde{\sigma}$ depending on ϵ, σ, p and d so that*

- (i) η is good for σ .
- (ii) $\tilde{\sigma} \subset \sigma$ and $\tilde{\sigma} \cap \mathcal{N}_\eta(\partial\sigma) = \emptyset$.
- (iii) There are positive constants $c_1(p, d, \epsilon, \sigma)$ and $c_2(p, d, \epsilon, \sigma)$ so that $h \leq \eta$ implies

$$\mathbb{P}_p(\mathfrak{X}_{\text{hemi}}(\tilde{\sigma}, h, r) \geq (1 + \epsilon)\mathcal{H}^{d-1}(r\sigma)\beta_{p,d}(v_\sigma)) \leq c_1 \exp(-c_2 r^{(d-1)/3})$$

Proof. Given a Borel set $E \subset \mathbb{R}^d$, and $a > 0$, we define the closed ℓ^1 -neighborhood $\mathcal{N}_a^{(1)}(E)$ to be all points in \mathbb{R}^d having an ℓ^1 -distance less than or equal to a from the set E .

Let $\epsilon' > 0$, let $\eta = \eta(\epsilon', \sigma) > 0$ be small enough so that it is good for σ and let $\tilde{\sigma}$ be the closure of $\sigma \setminus \mathcal{N}_\eta^{(1)}(\partial\sigma)$. Then $\tilde{\sigma}$ is a $(d-1)$ polytope, and because σ is connected (see the definition of a connected polytope given at the beginning of the present subsection), we may choose η sufficiently small so that $\tilde{\sigma}$ is also connected, and so that

$$(5.17) \quad \mathcal{H}^{d-1}(\sigma \setminus \tilde{\sigma}) \leq \epsilon' \mathcal{H}^{d-1}(\sigma)$$

As a consequence of this construction, we also have that $\tilde{\sigma} \subset \sigma$ and $\tilde{\sigma} \cap \mathcal{N}_\eta(\partial\sigma) = \emptyset$, so that (i) and (ii) are satisfied.

To show that (iii) holds, we use the same strategy as in the proof of Proposition 5.2. Let $h \leq \eta$, and let $\tilde{\sigma}' \subset \mathbb{R}^{d-1}$ be a $(d-1)$ -polytope with an isometry $\varphi : \tilde{\sigma}' \rightarrow \tilde{\sigma}$. Choose $k \in \mathbb{N}$ large enough so that $2^{-k} < h$, and large enough so that

$$(5.18) \quad \mathcal{L}^{d-1} \left(\tilde{\sigma}' \setminus \bigcup_{S' \in \Delta^k, S' \subset \tilde{\sigma}'} S' \right) \leq \epsilon' \mathcal{L}^{d-1}(\tilde{\sigma}')$$

Where, as before, Δ^k denotes the collection of dyadic squares in \mathbb{R}^{d-1} at scale k . We enumerate such squares contained in $\tilde{\sigma}'$ as S'_1, \dots, S'_m . Let $\delta > 0$ and dilate each S'_i about its center by a factor of $(1 - \delta)$ to produce a new, disjoint collection $\{S''_i\}_{i=1}^m$ of closed squares contained in $\tilde{\sigma}'$. Let $S_i = \varphi(S''_i)$, and choose δ small enough so that

$$(5.19) \quad \mathcal{H}^{d-1} \left(\tilde{\sigma} \setminus \bigcup_{i=1}^m S_i \right) < 2\epsilon' \mathcal{H}^{d-1}(\sigma)$$

We let $\alpha = (1 - \delta)2^{-k}$ as before, and we may assume that $\tilde{\sigma}'$ and φ are compatible with the chosen rotation, so that each S_i is $\alpha S(y_i, v_\sigma)$ for some $y_i \in \mathbb{R}^d$.

For each i and $r > 0$, let $E_r^{(i)}(\omega)$ denote a cutset in $\text{d-cyl}(S_i, \alpha, r)$ separating $\text{d-hemi}^\pm(S_i, \alpha, r)$ so that within the configuration ω , we have $|E_r^{(i)}(\omega)|_\omega = \mathfrak{X}_{\text{hemi}}(S_i, \alpha, r)$. Let A_r denote the collection of edges having non-empty intersection with

$$(5.20) \quad \mathcal{N}_{5d} \left(r \left(\tilde{\sigma} \setminus \bigcup_{i=1}^m S_i \right) \right)$$

so that $|A_r| \leq c(d)\epsilon' \mathcal{H}^{d-1}(r\sigma)$. The standard argument from the proof of Lemma 3.1 and throughout Section 3 tells us the edges of $A_r \cup \bigcup_{i=1}^m E_r^{(i)}(\omega)$ which are contained in $\text{d-cyl}(\tilde{\sigma}, h, r)$ separate the hemispheres of $\text{d-hemi}^\pm(\tilde{\sigma}, h, r)$. Here, we are also using the fact that we chose k large enough so that $\alpha \leq 2^{-k} \leq h$. We conclude that

$$(5.21) \quad \mathfrak{X}_{\text{hemi}}(\tilde{\sigma}, h, r) \leq c(d)\epsilon' \mathcal{H}^{d-1}(r\sigma) + \sum_{i=1}^m \mathfrak{X}_{\text{hemi}}(S_i, \alpha, r)$$

Thus,

$$(5.22) \quad \mathbb{P}_p(\mathfrak{X}_{\text{hemi}}(\tilde{\sigma}, h, r) \geq (1 + \epsilon)\mathcal{H}^{d-1}(r\sigma)\beta_{p,d}(v_\sigma))$$

$$(5.23) \quad \leq \mathbb{P}_p \left(\sum_{i=1}^m \mathfrak{X}_{\text{hemi}}(S_i, \alpha, r) \geq (1 + \epsilon - c(p, d)\epsilon')\mathcal{H}^{d-1}(r\sigma)\beta_{p,d}(v_\sigma) \right)$$

We now choose ϵ' small enough depending on p, d, ϵ so that $1 + \epsilon - c(p, d)\epsilon' \geq 1 + \epsilon/2$.

$$(5.24) \quad \mathbb{P}_p(\mathfrak{X}_{\text{hemi}}(\tilde{\sigma}, h, r) \geq (1 + \epsilon)\mathcal{H}^{d-1}(r\sigma)\beta_{p,d}(v_\sigma))$$

$$(5.25) \quad \leq \mathbb{P}_p \left(\sum_{i=1}^m \mathfrak{X}_{\text{hemi}}(S_i, \alpha, r) \geq (1 + \epsilon/2) \sum_{i=1}^m \mathcal{H}^{d-1}(rS_i)\beta_{p,d}(v_\sigma) \right)$$

$$(5.26) \quad \leq \sum_{i=1}^m \mathbb{P}_p \left(\mathfrak{X}_{\text{hemi}}(S_i, \alpha, r) \geq (1 + \epsilon/2)\mathcal{H}^{d-1}(rS_i)\beta_{p,d}(v_\sigma) \right)$$

$$(5.27) \quad \leq \sum_{i=1}^m \mathbb{P}_p \left(\mathfrak{X}(y_i, v_\sigma, \alpha r) \geq (1 + \epsilon/2)\mathcal{H}^{d-1}(\alpha r S(y_i, v_\sigma))\beta_{p,d}(v_\sigma) \right)$$

where we have used a union bound and the fact that each $S_i = \alpha S(y_i, v_\sigma)$ for some $y_i \in \mathbb{R}^d$. We apply our concentration estimates, Theorem 4.6, to each summand on the right, and we obtain

$$(5.28) \quad \mathbb{P}(\mathfrak{X}_{\text{hemi}}(\tilde{\sigma}, h, r) \geq (1 + \epsilon)\mathcal{H}^{d-1}(r\sigma)\beta_p(v_\sigma)) \leq mc_1 \exp(-c_2(\alpha r)^{(d-1)/3})$$

which completes the proof. \square

We now use a d -polytope P to form a valid subgraph of \mathbf{C}_n , producing a high probability upper bound on $\widehat{\Phi}_n$ in terms of the surface energy and volume of P .

Theorem 5.5. *Let $d \geq 2$ and let $p > p_c(d)$. Let $P \subset [-1, 1]^d$ be a polytope such that $\mathcal{L}^d(P) \leq 2^d/d!$, and let $\epsilon > 0$. There exist positive constants $c_1(p, d, \epsilon, P)$ and $c_2(p, d, \epsilon, P)$ so that*

$$\mathbb{P}_p \left(\widehat{\Phi}_n \geq (1 + \epsilon) \left(\frac{\mathcal{I}_{p,d}(nP)}{\theta_p(d)\mathcal{L}^d(nP)} \right) \right) \leq c_1 \exp(-c_2 n^{(d-1)/3})$$

Proof. We begin by working with $P_\delta := (1 - \delta)P$ for $\delta > 0$; we will choose δ carefully towards the end of the argument.

Note that the Euclidean distance from P_δ to $\partial[-1, 1]^d$ is positive. Let $\epsilon, \epsilon' > 0$ and enumerate the faces of P_δ as $\sigma_1, \dots, \sigma_m$, suppressing the dependence of these faces on δ . We use Proposition 5.4, picking η

depending on ϵ' and on each face σ_i . For each i , we produce a $\tilde{\sigma}_i$ such that $\mathcal{H}^{d-1}(\sigma_i \setminus \tilde{\sigma}_i) \leq \epsilon' \mathcal{H}^{d-1}(\sigma_i)$ and so that

(i) η is good for σ_i .

(ii) $\tilde{\sigma}_i \subset \sigma_i$ and $\tilde{\sigma}_i \cap \mathcal{N}_\eta(\partial\sigma_i) = \emptyset$.

(iii) There are positive constants $c_1(p, d, \epsilon', P, \delta)$ and $c_2(p, d, \epsilon', P, \delta)$ so that if $h \leq \eta$,

$$(5.29) \quad \mathbb{P}_p(\mathfrak{X}_{\text{hemi}}(\tilde{\sigma}_i, h, r) \geq (1 + \epsilon') \mathcal{H}^{d-1}(r\sigma_i) \beta_{p,d}(v_\sigma)) \leq c_1 \exp(-c_2 r^{(d-1)/3})$$

We may assume that η is small enough so that the closed neighborhood $\mathcal{N}_\eta(P_\delta) \subset [-1, 1]^d$, and so that η is good for P_δ itself. Let Q_1, \dots, Q_ℓ enumerate the dyadic cubes at scale k within $[-1, 1]^d$. We suppose these cubes are ordered so that for $\ell_1 \leq \ell_2 \in \{1, \dots, \ell\}$, we have that Q_1, \dots, Q_{ℓ_2} enumerates all such cubes having non-empty intersection with $\mathcal{N}_\eta(P_\delta)$, and also that Q_1, \dots, Q_{ℓ_1} enumerates all such cubes contained within $P_\delta \setminus \mathcal{N}_\eta(P_\delta)$. We take k sufficiently large and take η smaller if necessary so that

$$(5.30) \quad \mathcal{L}^d \left(\bigcup_{j=\ell_1+1}^{\ell_2} Q_j \right) < \epsilon' \mathcal{L}^d(P_\delta)$$

For each $j \in \{1, \dots, \ell\}$, let $\mathcal{E}_n^{(j)}$ be the event that

$$(5.31) \quad \left\{ \frac{|\mathbf{C}_\infty \cap nQ_j|}{\mathcal{L}^d(nQ_j)} \in (\theta_p(d) - \epsilon', \theta_p(d) + \epsilon') \right\}$$

For each face σ_i , let $E_n^{(i)}(\omega)$ denote a cutset within $d\text{-cyl}(\tilde{\sigma}_i, h, n)$ separating $d\text{-hemi}^\pm(\tilde{\sigma}_i, h, n)$, and such that $|E_n^{(i)}|_\omega$ is $\mathfrak{X}_{\text{hemi}}(\tilde{\sigma}_i, h, n)$ in the configuration ω . We let A_n be the collection of edges having non-empty intersection with

$$(5.32) \quad \mathcal{N}_{5d} \left(n \left(\partial P_\delta \setminus \bigcup_{i=1}^m \tilde{\sigma}_i \right) \right)$$

By our choice of the $\tilde{\sigma}_i$, we have that $|A_n| \leq c(d)\epsilon' \mathcal{H}^{d-1}(\partial P_\delta) n^{d-1}$. We define Γ_n to be the following set of edges:

$$(5.33) \quad \Gamma_n := \left(\bigcup_{i=1}^m E_i^{(n)}(\omega) \right) \cup A_n$$

We now define the vertex set H_n to be all vertices $v \in \mathbf{C}_n$ such that any path from v to ∞ in \mathbf{C}_∞ must use an edge of Γ_n . The proof of Lemma 3.1 tells us that not only is H_n non-empty, but that it also contains every vertex $v \in \mathbf{C}_\infty \cap Q_j$ for $j \in \{1, \dots, \ell_1\}$. We work within the intersection of the high probability events $\mathcal{E}_n^{(j)}$ to obtain control on the volume of H_n . Within this event we have:

$$(5.34) \quad (\theta_p(d) - \epsilon') \left(\sum_{j=1}^{\ell_1} \mathcal{L}^d(nQ_j) \right) - \ell c(d)(2^{-k}n)^{d-1} \leq |H_n| \leq (\theta_p(d) + \epsilon') \sum_{j=1}^{\ell_2} \mathcal{L}^d(nQ_j)$$

where the term we subtract on the left comes from the fact that the Q_j are not disjoint, but rather have disjoint interiors. For n sufficiently large (depending on p, d, ϵ', P), we have

$$(5.35) \quad (\theta_p(d) - 2\epsilon') \sum_{j=1}^{\ell_1} \mathcal{L}^d(nQ_j) \leq |H_n| \leq (\theta_p(d) + \epsilon') \sum_{j=1}^{\ell_2} \mathcal{L}^d(nQ_j)$$

and hence that

$$(5.36) \quad (\theta_p(d) - 2\epsilon')(1 - \epsilon') \mathcal{L}^d(nP_\delta) \leq |H_n| \leq (\theta_p(d) + \epsilon')(1 + \epsilon') \mathcal{L}^d(nP_\delta)$$

$$(5.37) \quad (\theta_p(d) - 2\epsilon')(1 - \epsilon')(1 - \delta)^d \mathcal{L}^d(nP) \leq |H_n| \leq (\theta_p(d) + \epsilon')(1 + \epsilon')(1 - \delta)^d \mathcal{L}^d(nP)$$

We now show that H_n is a valid subgraph of \mathbf{C}_n when δ is chosen appropriately. On the intersection of the events $\{\mathcal{E}_j^{(n)}\}_{j=1}^\ell$, and due to the fact that the cubes Q_j intersect one another at their boundaries, we have

$$(5.38) \quad |\mathbf{C}_n| \geq (\theta_p(d) - \epsilon')(2n)^d - \ell c(d)(2^{-k}n)^{d-1}$$

$$(5.39) \quad \geq (\theta_p(d) - 2\epsilon')(2n)^d$$

for n sufficiently large. As $\mathcal{L}^d(P) \leq 2^d/d!$, choosing δ so that

$$(5.40) \quad (\theta_p(d) + \epsilon')(1 + \epsilon')(1 - \delta)^d = (\theta_p(d) - 2\epsilon')$$

ensures that H_n is a valid subgraph of \mathbf{C}_n . Note that defining δ this way means that $\delta \rightarrow 0$ as $\epsilon' \rightarrow 0$. Not only have we shown that H_n is valid, we also have a lower bound on its volume. To obtain a bound on $\widehat{\Phi}_n$ then, it suffices to find an upper bound on $\partial^\omega H_n$. In fact, from the construction of H_n , we have $\partial^\omega H_n \subset \Gamma_n$, so that

$$(5.41) \quad |\partial^\omega H_n| \leq \sum_{i=1}^m |E_n^{(i)}(\omega)|_\omega + c(d)\mathcal{H}^{d-1}(\partial P_\delta)\epsilon' n^{d-1}$$

For $i \in \{1, \dots, m\}$, let $\mathcal{F}_n^{(i)}$ be the following high probability event from Proposition 5.4:

$$(5.42) \quad \{\mathfrak{X}_{\text{face}}(\tilde{\sigma}_i, \eta, n) < (1 + \epsilon')\mathcal{H}^{d-1}(n\sigma_i)\beta(v_\sigma)\}$$

On the intersection of the events $\mathcal{F}_n^{(i)}$, we have

$$(5.43) \quad |\partial^\omega H_n| \leq (1 + \epsilon')\mathcal{I}_{p,d}(nP) + c(d)\mathcal{H}^{d-1}(\partial P)\epsilon' n^{d-1}$$

$$(5.44) \quad \leq (1 + \epsilon' + c(p, d)\epsilon')\mathcal{I}_{p,d}(nP)$$

Thus, on the intersection of all $\mathcal{E}_n^{(j)}$ and all $\mathcal{F}_n^{(i)}$, we have

$$(5.45) \quad \widehat{\Phi}_n \leq \left(\frac{1 + \epsilon' + c(p, d)\epsilon'}{(\theta_p(d) - 2\epsilon')(1 - \epsilon')(1 - \delta)^d} \right) \frac{\mathcal{I}_{p,d}(nP)}{\mathcal{L}^d(nP)}$$

We take ϵ' small enough (recall $\delta = \delta(\epsilon')$ goes to zero as ϵ' does) so that

$$(5.46) \quad \widehat{\Phi}_n \leq (1 + \epsilon) \frac{\mathcal{I}_{p,d}(nP)}{\mathcal{L}^d(nP)}$$

We use the bounds in Proposition 5.4 and in Corollary A.4 to conclude that

$$(5.47) \quad \mathbb{P}_p \left(\left(\bigcap_{i=1}^m \mathcal{F}_n^{(i)} \cap \bigcap_{j=1}^\ell \mathcal{E}_n^{(j)} \right)^c \right) \leq mc_1 \exp(-c_2 n^{(d-1)/3}) + \ell c_1 \exp(-c_2 (2^{-k}n)^{d-1})$$

which completes the proof, upon tracking the dependencies of ℓ and k . \square

Using Proposition A.14 and Borel-Cantelli, we extract from Theorem 5.5 a statement involving the Wulff crystal $W_{p,d}$.

Corollary 5.6. *Consider the Wulff crystal $W_{p,d}$ corresponding to the norm $\beta_{p,d}$, and let $\epsilon > 0$. The event*

$$\left\{ \limsup_{n \rightarrow \infty} n\widehat{\Phi}_n \leq (1 + \epsilon) \frac{\mathcal{I}_{p,d}(W_{p,d})}{\theta_p(d)\mathcal{L}^d(W_{p,d})} \right\}$$

occurs \mathbb{P}_p -almost surely.

Proof. Recall that $\mathcal{L}^d(W_{p,d}) = 2^d/d!$. Let $\epsilon, \epsilon' > 0$ and apply Proposition A.14 with this parameter to obtain $P_{\epsilon'} \subset W_{p,d}$ with $|\mathcal{I}_{p,d}(P_{\epsilon'}) - \mathcal{I}_{p,d}(W_{p,d})| < \epsilon'$ and with $\mathcal{L}^d(W_{p,d} \setminus P_{\epsilon'}) < \epsilon'$. Apply Theorem 5.5 to the polytope $P_{\epsilon'}$ to obtain positive constants $c_1(p, d, \epsilon, P_{\epsilon'})$ and $c_2(p, d, \epsilon, P_{\epsilon'})$ so that

$$(5.48) \quad \mathbb{P}_p \left(n\widehat{\Phi}_n \geq (1 + \epsilon/2) \left(\frac{\mathcal{I}_{p,d}(P_{\epsilon'})}{\theta_p(d)\mathcal{L}^d(P_{\epsilon'})} \right) \right) \leq c_1 \exp(-c_2 n^{(d-1)/3})$$

Thus,

$$(5.49) \quad \mathbb{P}_p \left(n\widehat{\Phi}_n \geq (1 + \epsilon/2) \left(\frac{1 + \epsilon'}{1 - \epsilon'} \right) \left(\frac{\mathcal{I}_{p,d}(W_{p,d})}{\theta_p(d)\mathcal{L}^d(W_{p,d})} \right) \right) \leq c_1 \exp(-c_2 n^{(d-1)/3})$$

Choosing ϵ' sufficiently small depending on ϵ and applying Borel-Cantelli completes the proof. \square

Corollary 5.6 is a large step towards Theorem 1.3. Before moving to the next section, let us make an observation that will facilitate the proof of Theorem 1.2. For $W \subset [-1, 1]^d$ a translate of the Wulff crystal, we define the *empirical measure* associated to W as

$$(5.50) \quad \bar{\nu}_W(n) := \frac{1}{n^d} \sum_{x \in \mathbf{C}_n \cap nW} \delta_{x/n}$$

From the proofs of Theorem 5.5 and Corollary 5.6, we may deduce the following.

Corollary 5.7. *Let $d \geq 2$, $p > p_c(d)$ and let $W \subset [-1, 1]^d$ be a translate of the Wulff crystal $W_{p,d}$. For $\epsilon > 0$, there are positive constants $c_1(p, d, \epsilon)$ and $c_2(p, d, \epsilon)$ so that*

$$\mathbb{P}_p(\mathfrak{d}(\bar{\nu}_W(n), \nu_W) > \epsilon) \leq c_1 \exp(-c_2 n^{d-1})$$

Indeed, this corollary follows from the density result Corollary A.4 used in the proof of Theorem 5.5, the approximation of the Wulff crystal by polytopes (Proposition A.14) and the definition of the metric \mathfrak{d} . No concentration estimates are needed, but the proof of Corollary 5.7 is so similar to that of Theorem 5.5 that we find it convenient to state it here. The remainder of the paper is devoted to showing the bound from Corollary 5.6 cannot be improved.

6. COARSE GRAINING

Having spent the last two sections passing from continuous objects to discrete objects, we now move in the other, more difficult direction. For each Cheeger optimizer $G_n \in \mathcal{G}_n$, we would like to produce a corresponding Borel set $P_n \subset [-1, 1]^d$ so that the conductance of nP_n is comparable to that of G_n . Intuitively, this should imply the sequence $\text{per}(P_n)$ is uniformly bounded in n , with Lemma A.10 making this intuition rigorous.

A natural first choice for the P_n are the sets mentioned in the introduction, built by replacing each vertex of a Cheeger optimizer with its corresponding dual unit cube. The issue with this choice is that the perimeter of these polytopes is directly related to ∂G_n instead of $\partial^\omega G_n$. While results like Lemma A.10 provide us with control on $\partial^\omega G_n$, we have much less control on ∂G_n unless p is very close to one. This suggests a renormalization argument, and indeed, we will use an involved renormalization argument of Zhang to “tame” the full edge boundary of G_n .

As an intermediate step towards producing a sequence P_n , we first augment each $G_n \in \mathcal{G}_n$ to some $F_n \subset \mathbf{C}_n$. Each F_n will be built in a way that allows us to control its boundary. Moreover, to each F_n , we may associate an empirical measure $\tilde{\mu}_n$ as in Section 2, and we will show that $\tilde{\mu}_n$ and μ_n are \mathfrak{d} -close. The goal of the present section is to construct such an F_n for each G_n , and to also build corresponding edge sets Γ_n , where Γ_n should be thought of as the boundary of F_n and has cardinality on the order of n^{d-1} .

That each Γ_n is surface order will enable us to construct a suitable set P_n from F_n , and we do this in Section 7. The endgame of this construction is that, with high probability, each μ_n will be \mathfrak{d} -close to some

ν_F repressing a set of finite perimeter at most some γ , where γ does not depend on n . In Section 8, we show that when a given μ_n is \mathfrak{d} -close to such a measure ν_F , we will be able to relate the conductance of G_n to that of F by working locally on the boundary of F . Corollary 5.6 will tell us that unless F is the Wulff crystal, it should be impossible for μ_n to be too \mathfrak{d} -close to F . From this reasoning, we will obtain the main theorems of the paper.

One artifact of the construction given in this section is that we will need to now restrict our attention to $d \geq 3$. We will comment more on this complication in Section 6.2, but for now we simply assume $d \geq 3$ for the remainder of the paper.

6.1. The construction of Zhang. Let k be a natural number which we refer to as the *renormalization parameter*. Given $x \in \mathbb{Z}^d$, we define the k -cube corresponding to x as:

$$(6.1) \quad \underline{B}(x) = (2k)x + [-k, k]^d$$

We suppress the dependence of $\underline{B}(x)$ on k to avoid cumbersome notation. We use an underscore to denote sets of k -cubes. If \underline{G} is a set of k -cubes and if $x \in \mathbb{Z}^d$ is a vertex, we write $x \in \underline{G}$ if x is contained in one of the k -cubes of \underline{G} . If $e \in E(\mathbb{Z}^d)$ is an edge, we write $e \in \underline{G}$ if both endpoint vertices of e lie in \underline{G} .

We also need to introduce a type of larger cube; we define a $3k$ -cube $\underline{B}_3(x)$ as follows:

$$(6.2) \quad \underline{B}_3(x) = (2k)x + [-3k, 3k]^d$$

We emphasize that x must lie in \mathbb{Z}^d , so that each $3k$ -cube contains exactly 3^d k -cubes. Two cubes $\underline{B}(x)$ and $\underline{B}(x')$ are *adjacent* if $x \sim x'$, or equivalently if they share a face. Two cubes $\underline{B}(x)$ and $\underline{B}(x')$ are \mathbb{L}^d -adjacent if $x \sim_{\mathbb{L}} x'$, or equivalently if either $\underline{B}(x') \subset \underline{B}_3(x)$ or $\underline{B}(x) \subset \underline{B}_3(x')$.

We now follow a construction of Zhang from Section 2 of [67]. We describe Zhang's method in general first, and then apply it to G_n . The idea is to form a collection of k -cubes which contain $\partial_o G_n$, and then to discover within this collection another, more tame cutset separating G_n and ∞ .

Let $G \subset \mathbb{C}_\infty$ be a finite, connected graph. The graph $G = G(\omega)$ is allowed to depend on the percolation configuration. From G , we define several sets of k -cubes. Firstly, we define

$$(6.3) \quad \underline{G} := \{\underline{B}(x) : \underline{B}(x) \cap (G \cup \partial_o G) \neq \emptyset\} \quad \text{and} \quad \underline{A} := \{\underline{B}(x) : \underline{B}(x) \cap \partial_o G \neq \emptyset\}$$

Figure 5 depicts a possible G as well as the k -cube set \underline{A} . As G is finite, so too is \underline{G} , thus the cubes $\underline{B}(x)$ which are not in \underline{G} split into a single infinite \mathbb{L}^d -connected component, which we label \underline{Q} , as well as finitely many finite \mathbb{L}^d -connected components $\underline{Q}'(1), \dots, \underline{Q}'(u')$. We refer to \underline{Q} as the *ocean* and following Zhang's terminology, we refer to the $\underline{Q}'(i)$ as *ponds*.

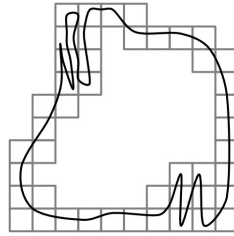


FIGURE 5. The black line and its interior represent $\partial_o G$ and G respectively. Notice that \underline{A} , depicted by the squares covering $\partial_o G$, is not necessarily the boundary of \underline{G} .

For the ocean \underline{Q} , we use $\Delta \underline{Q}$ to denote the set of k -cubes $\underline{B}(x)$ which are \mathbb{L}^d -adjacent to a cube in \underline{Q} , but are not themselves contained in \underline{Q} . Likewise, for each pond $\underline{Q}'(i)$, we let $\Delta \underline{Q}'(i)$ denote the cubes \mathbb{L}^d -adjacent to $\underline{Q}'(i)$ but not contained in $\underline{Q}'(i)$.

The next step in Zhang's construction is to pass to the unique configuration ω' obtained by closing each open edge in $\partial_o G$. We do this while preserving both G and \mathbf{C}_∞ . In other words, when we work in the configuration ω' , we still work with $G(\omega)$ and $\mathbf{C}_\infty(\omega)$, but with each open edge of $\partial_o G$ now closed. Somewhat paradoxically then, \mathbf{C}_∞ may be a disconnected graph after passing to the configuration ω' . We hope this notation does not generate confusion, and we emphasize that \mathbf{C}_∞ below is *not* $\mathbf{C}_\infty(\omega')$.

We now pass to the configuration ω' . Each pond may intersect an open cluster which is connected to the ocean. Importantly, these open clusters do not need to be contained in \mathbf{C}_∞ . We say a pond is *live* if it intersects an open cluster which also intersects the ocean. We say a pond is *almost-live* if it intersects an open cluster which also intersects a live pond. We say a pond is *dead* if it is neither live nor almost-live. We refine the collection of ponds $\{\underline{Q}'(i)\}_{i=1}^{u'}$ to the collection of live and almost-live ponds $\underline{Q}(1), \dots, \underline{Q}(u)$. Figure 6 depicts a possible configuration of ponds. Zhang uses the terminology “live” and “dead” in [67], and we find it necessary for the argument to introduce the intermediate “almost-live” status.¹

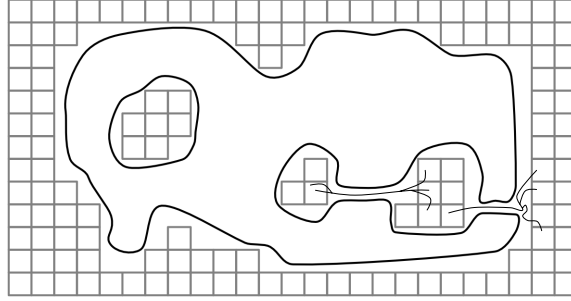


FIGURE 6. The graph G is represented by the closed curve. The connected components of cubes in the diagram are ponds or the ocean. The left-most pond is dead, the right-most pond is live and the middle pond is almost-live.

Let C denote the collection of all open clusters which intersect \underline{Q} , and let C_i denote the collection of all open clusters which intersect the live or almost-live pond $\underline{Q}(i)$. We emphasize again that the components of C and C_i are not necessarily in \mathbf{C}_∞ . To isolate the vertices of C_i within the $\underline{Q}(i)$ only, we define $Q_i := C_i \cap \underline{Q}(i)$, and we likewise define $Q := C \cap \underline{Q}$. We let Bridge be the remainder of these components which lie in \underline{G} . Specifically,

$$(6.4) \quad \text{Bridge} := \left[\left(\bigcup_{\underline{B}(x) \in \underline{G}} \underline{B}(x) \right) \cap \left(C \cup \left(\bigcup_{i=1}^u C_i \right) \right) \right] \setminus \left(Q \cup \left(\bigcup_{i=1}^u Q_i \right) \right)$$

One can refer to Figure 6 for an example of the set Bridge; we need only excise the portions of the thin black lines intersecting any cubes drawn. Though Bridge has been defined as a set of vertices, it is equipped with an obvious graph structure inherited from C and the C_i .

Let us make a few observations concerning the live and almost-live ponds as well as Bridge. The first observation is that ponds may be used to \mathbb{Z}^d -paths connecting to G .

Lemma 6.1. *Pass to the configuration ω' in which each open edge of $\partial_o G$ is made closed, and suppose γ is a simple \mathbb{Z}^d -path which begins from a vertex in some $\underline{Q}(i)$, and which ends at a vertex of G . Suppose also that γ is contained in $\underline{Q}(i) \cup \Delta \underline{Q}(i)$. Then γ must use a closed edge in $\Delta \underline{Q}(i)$.*

Proof. Note that by construction, any live or almost-live pond is joined to ∞ by a \mathbb{Z}^d -path not using any vertex of G . Within the configuration ω' , G is necessarily separated from ∞ by a closed cutset.

¹There was a strong temptation to use the term “mostly-dead,” in honor of the 1987 film *The Princess Bride*, but we ultimately felt “almost-live” to be more intuitive.

As $\underline{Q}(i) \cap G = \emptyset$, if γ is connected to G , it must leave $\underline{Q}(i)$ at some point. Let v_1 be the last vertex of $\underline{Q}(i)$ used by γ , and let γ_1 denote the truncation of γ started at v_1 . As $v_1 \in \underline{Q}(i)$, we see that v_1 is joined to ∞ by a \mathbb{Z}^d path which does not use any vertex of G . Thus γ_1 necessarily uses a closed edge. Because v_1 is the only vertex of γ_1 in $\underline{Q}(i)$, and because γ is contained in $\underline{Q}(i) \cup \Delta\underline{Q}(i)$, this closed edge must necessarily be contained in $\Delta\underline{Q}(i)$. \square

The second observation we make concerns Bridge, with graph structure as specified above.

Lemma 6.2. *In the configuration ω' , the vertex sets of Bridge and G are disjoint, and all edges of ∂Bridge are closed, except for those joining a vertex of Bridge and a vertex of some Q_i , or an edge joining a vertex of Bridge with a vertex in Q .*

Proof. To show Bridge and G are disjoint, it suffices to show for each i that $C_i \cap G = \emptyset$, and that $C \cap G = \emptyset$. Were $C \cap G \neq \emptyset$, we would have $\underline{Q} \cap G \neq \emptyset$, which is impossible. The same reasoning shows $C_i \cap G = \emptyset$.

Because C and the C_i are themselves collections of open clusters, it is impossible for ∂C or ∂C_i to contain open edges. Upon reviewing the definition of Bridge, we see the only open edges of ∂Bridge must extend either into the ocean \underline{Q} or to some live or almost-live pond $\underline{Q}(i)$. We have not used that we are in the configuration ω' , except implicitly through the fact that C and C_i are only defined after passing from ω to ω' . \square

We define the set of k -cubes associated to Bridge as

$$(6.5) \quad \underline{\text{Bridge}} := \{B(x) : B(x) \text{ contains a vertex of Bridge}\}$$

and we finally construct the central cube set of Zhang's argument, which is depicted in Figure 7. Let

$$(6.6) \quad \underline{\Gamma} := \Delta\underline{Q} \cup \underline{\text{Bridge}} \cup \left(\bigcup_{i=1}^u \Delta\underline{Q}(i) \right)$$

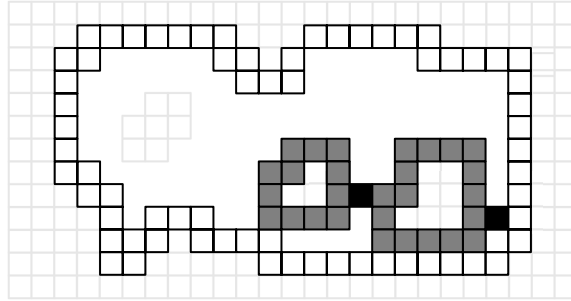


FIGURE 7. We have removed $\partial_o G$ and Bridge from the diagram for the sake of clarity, but this picture is built from Figure 6. The white cubes depict $\Delta\underline{Q}$, the grey cubes depict the two $\Delta\underline{Q}(i)$ and the black cubes depict $\underline{\text{Bridge}}$. Actually, the cubes adjacent to the black cubes are also in $\underline{\text{Bridge}}$, so this tells us that $\underline{\text{Bridge}}$ is not necessarily disjoint from the boundary of the ponds and ocean.

For each k -cube $B(x)$, we can define the corresponding *augmented* cube $B^+(x)$ as

$$(6.7) \quad B^+(x) = 2kx + [-2k - 1, 2k + 1]^d$$

We define the augmented version of $\underline{\Gamma}$ as $\underline{\Gamma}^+ := \{B^+(x) : B(x) \in \underline{\Gamma}\}$.

Proposition 6.3. *In the configuration ω' , the augmented cube set $\underline{\Gamma}^+$ contains a closed cutset Γ which separates G from ∞ .*

Proof. Let γ' be a path from G to ∞ . We will show that γ' uses a closed edge which is contained in Γ^+ . We lose no generality supposing γ' is simple. As \underline{G} and each of the ponds are finite, it must be that γ' eventually lies entirely within the ocean \underline{Q} . Let us pick a vertex v_0 within \underline{Q} such that γ' uses x_0 , and every vertex used by γ' after x_0 is also within \underline{Q} . We cut γ' off at x_0 , and reverse this path so that it is now a path from x_0 to G . We denote this truncated and then reversed path as γ . It will suffice to show that γ uses a closed edge which is contained in Γ^+ .

As \underline{Q} does not intersect G , it must be that γ uses a last vertex contained in \underline{Q} , which we denote as v_1 . We denote the truncation of γ started at v_1 as γ_1 . Based on our careful construction of \underline{Q} and the \underline{Q}_i , it is possible that $v_1 \in \text{Bridge}$ despite the fact that $v_1 \in \underline{Q}$.

If the edge following v_1 in γ_1 is closed, we are content as this edge lies in $\Delta\underline{Q}$. Thus we may suppose that the edge following v_1 is open, so that γ_1 joins the vertex set Bridge . The path γ_1 must connect with G . By Lemma 6.2, Bridge and G are disjoint. There are two cases: in the first case, γ_1 first leaves Bridge through an open edge. In the second, γ_1 first leaves Bridge through a closed edge. In the second case, the closed edge used by γ_1 is contained in one of the augmented cubes of Bridge , so this is handled entirely.

Back in the first case, that γ_1 leaves Bridge through an open edge implies by Lemma 6.2 it must pass into some \underline{Q}_i . But, for γ_1 to reach G , it must exit $\underline{Q}(i)$. Let v_2 denote the last vertex in $\underline{Q}(i)$ used by the path γ_1 . Let γ_2 be the remainder of the path γ_1 beginning from v_2 .

There are two sub-cases. In sub-case (a), γ_2 is contained in $\underline{Q}(i) \cup \Delta\underline{Q}(i)$. In sub-case (b), γ_2 is not contained in $\underline{Q}(i) \cup \Delta\underline{Q}(i)$. Lemma 6.1 tells us that in case (a), γ_2 must use a closed edge contained in $\Delta\underline{Q}(i)$. Thus we may suppose sub-case (b) holds. If the edge e incident to v_2 in the path γ_2 is closed, our claim holds, as this edge must lie in $\Delta\underline{Q}(i)$. On the other hand, if e is open, the path γ_2 has rejoined the set Bridge , and can only exit Bridge through an open edge if it passes into another pond $\underline{Q}(j)$. This new pond must be distinct from $\underline{Q}(i)$.

There are finitely many ponds, so the finiteness of γ implies that we must either find ourselves back in the sub-case (a) for a different live or almost-live pond, or we must find ourselves in the following unique situation: there is a vertex v_3 which is the last vertex in any pond $\underline{Q}(i)$ used by γ_2 , but the truncation γ_3 of γ_2 started at v_3 is not contained within $\underline{Q}(i) \cup \Delta\underline{Q}(i)$ for any live or almost-live pond $\underline{Q}(i)$. If the edge immediately following v_3 is closed, our claim holds as this edge will lie in $\Delta\underline{Q}(i)$ for some live or almost-live pond. If it is open, γ_3 has rejoined Bridge , but can now only exit Bridge through a closed edge as it cannot return to \underline{Q} or any $\underline{Q}(i)$.

This shows Γ^+ contains a closed cutset separating G from ∞ . Using for instance, the same method used to pick a unique G_n in each configuration, we choose the closed minimal cutset in Γ^+ in some unique way and label it Γ . \square

Next, we show that Γ is contained in the coarse-grained image of $\partial_o G$.

Lemma 6.4. *The cube set Γ is contained in \underline{A} .*

Proof. Suppose that $\underline{B}(x) \in \text{Bridge}$. We claim that $\underline{B}(x)$ must contain either a vertex of G or an endpoint vertex of $\partial_o G$. If not, then $\underline{B}(x)$ must be a member of some pond $\underline{Q}'(i)$ or of the ocean \underline{Q} . But that $\underline{B}(x) \in \text{Bridge}$ implies there is $y \in \underline{B}(x)$ which lies within an open cluster connected to a live or almost-live pond. Thus, if $\underline{B}(x)$ is a member of a pond, this pond must be live or almost-live. This implies that $y \in \underline{Q}_i$ for some $i \in \{1, \dots, u\}$ or $y \in \underline{Q}$. This is impossible, as we cut out such vertices in the construction of Bridge .

Thus, if $\underline{B}(x) \in \text{Bridge}$, it must be that $\underline{B}(x)$ contains either a vertex of G or an endpoint vertex of $\partial_o G$. As $\underline{B}(x)$ contains some $y \in \text{Bridge}$, and as this y lies within an open cluster contained in C or some C_i which is disjoint from G , any path γ from y to G within the box $\underline{B}(x)$ must use some edge e of ∂G . But for $y \in C$ or $y \in C_i$, we know that y is connected to ∞ via a path using no vertices of G . Thus the path γ from y to G within $\underline{B}(x)$ must actually use an edge of $\partial_o G$. This shows that $\text{Bridge} \subset \underline{A}$.

The next thing we need to show is that $\Delta Q(i) \subset \underline{A}$ for each live or almost live pond $Q(i)$. Let $\underline{B}(x) \in \Delta Q(i)$. Then $\underline{B}(x)$ is \mathbb{L}^d -adjacent to a cube $\underline{B}(x')$ in $Q(i)$, so $\underline{B}(x)$ must contain either a vertex of G or an endpoint vertex of $\partial_o G$, otherwise $\underline{B}(x)$ would be a member of $Q(i)$. We may thus suppose that $\underline{B}(x)$ contains a vertex y of G .

Note that $\underline{B}(x)$ and $\underline{B}(x')$ have at least one vertex z in common, and this vertex (by virtue of lying within some $Q(i)$) is connected to ∞ in via a \mathbb{Z}^d -path which does not use G . Any path joining y and z in $\underline{B}(x)$ must necessarily use an edge of $\partial_o G$. The final case of cubes within ΔQ is handled identically to the case of pond boundary cubes. \square

We now reflect on the connectivity of $\underline{\Gamma}$. We know from the work of Timár [63] (and earlier work of Kesten, Deuschel and Pisztora) that minimal cutsets separating G from ∞ are \mathbb{L}^d -connected, but $\underline{\Gamma}^+$ is not necessarily the coarse-grained image of the cutset Γ . Nevertheless, we can apply these results to the sets $\Delta Q(i)$ and ΔQ .

Lemma 6.5. *The k -cube set $\underline{\Gamma}$ is \mathbb{L}^d -connected.*

Proof. It follows directly from Lemma 2 of Timár [63] that ΔQ and each $\Delta Q(i)$ are \mathbb{L}^d -connected cube sets.

Let D be a connected component of Bridge, and let \underline{D} be the collection of k -cubes containing a vertex of D , so that $\underline{D} \subset \text{Bridge}$. It follows from the construction of Bridge that \underline{D} either intersects $\Delta Q(i)$ for some i , or \underline{D} intersects Q . As D is connected in \mathbb{Z}^d , it is immediate that coarse-grained image \underline{D} is \mathbb{L}^d -connected. Note that Bridge is the union of all such cube sets \underline{D} , and it follows from the defining properties of live and almost-live ponds that $\underline{\Gamma}$ is \mathbb{L}^d -connected. \square

A last and important aspect of Zhang's construction that each cube in $\underline{\Gamma}$ has a useful geometric property when G is sufficiently large. We introduce some more of Zhang's terminology. Each k -cube $\underline{B}(x)$ has $2d$ faces $\sigma_1(x), \dots, \sigma_{2d}(x)$, each of which an isometric image of $[-k, k]^{d-1} \subset \mathbb{R}^{d-1}$. We say that a *surface* of $\underline{B}(x)$ is a vertex set of the form $\sigma_i(x) \cap \mathbb{Z}^d$, so that each k -cube $\underline{B}(x)$ possesses $2d$ surfaces. A surface of a $3k$ -cube $\underline{B}_3(x)$ is just a surface of one of the k -cubes $\underline{B}(x') \subset \underline{B}_3(x)$.

Say that a k -cube $\underline{B}(x)$ is *Type-I* if there is an open path γ and a surface $\sigma \cap \mathbb{Z}^d$ in $\underline{B}_3(x)$ such that γ joins a vertex in $\underline{B}^+(x)$ to a vertex of $\partial \underline{B}_3(x) \cap \mathbb{Z}^d$ and such that no vertex along γ is joined via another open path to $\sigma \cap \mathbb{Z}^d$. We require that γ uses edges which are *internal* to $\underline{B}_3(x)$, that is, no edge of γ has both endpoints in $\partial \underline{B}_3(x)$. We also require any candidate for a path from a vertex of γ to $\sigma \cap \mathbb{Z}^d$ to use only edges internal to $\underline{B}_3(x)$.

We say a k -cube $\underline{B}(x)$ is *Type-II* if there are two open paths γ_1 and γ_2 , each of which connects a distinct vertex of $\underline{B}^+(x)$ to distinct vertices in $\partial \underline{B}_3(x)$, and such that there is no open path in $\underline{B}_3(x)$ joining any vertex in γ_1 to a vertex in γ_2 . We require that all paths in this definition use only edges internal to $\underline{B}_3(x)$. Figure 8 illustrates these two geometric properties.

Observe that, because of the requirement that all paths in the above definitions are internal, the event that a k -cube $\underline{B}(x)$ is Type-I or Type-II does not depend on any of the edges of \mathbb{Z}^d contained in $\partial \underline{B}_3(x)$.

Proposition 6.6. *Suppose that G is not contained within a $3k$ -cube. Then, in the configuration ω' , each k -cube of $\underline{\Gamma}$ is either Type-I or Type-II.*

Proof. Following Zhang, we consider two cases. In the first case, we suppose $\underline{B}(x) \in \underline{\Gamma}$ is a member of

$$(6.8) \quad \Delta Q \cup \left(\bigcup_{i=1}^u \Delta Q(i) \right)$$

Such a $\underline{B}(x)$ is \mathbb{L}^d -adjacent to a cube $\underline{B}(x')$ which neither intersects G nor an endpoint vertex of $\partial_o G$. Thanks to Lemma 6.4, we know $\underline{B}(x) \in \underline{A}$, so that $\underline{B}(x)$ contains an endpoint vertex of $\partial_o G$. Thus, $\underline{B}^+(x)$ contains a vertex $y \in G$. There can be no open path from any surface of $\underline{B}(x')$ to y . On the other hand, because

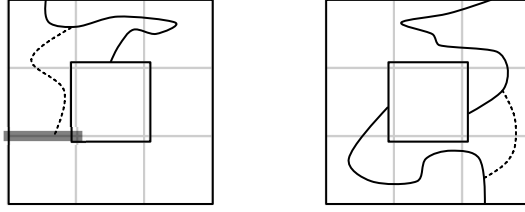


FIGURE 8. On the left, we see an illustration of what *cannot* happen in a Type-I cube. The dotted line is an open path joining the solid line (also an open path) to one of the surfaces of the $3k$ -cube. Likewise, on the right, we see an illustration of what *cannot* happen in a Type-II cube.

no connected component of G is contained in a $3k$ -cube, there must be an open path from y to a vertex of $\partial \underline{B}_3(x)$. We may arrange this open path uses edges internal to $\underline{B}_3(x)$ by stopping it at the first vertex of $\partial \underline{B}_3(x)$ it meets. Thus in the first case, $\underline{B}(x)$ is Type-I.

In the second case, we suppose

$$(6.9) \quad \underline{B}(x) \in \underline{\text{Bridge}} \setminus \left(\Delta \underline{Q} \cup \left(\bigcup_{i=1}^u \Delta \underline{Q}(i) \right) \right)$$

Let $y \in \underline{B}(x) \cap \text{Bridge}$. Then y lies in some connected component D of either C or one of the C_i . The component D cannot be contained in $\underline{B}_3(x)$, otherwise one of the k -cubes \mathbb{L}^d -adjacent to $\underline{B}(x)$ would be a member of either \underline{Q} or one of the $\underline{Q}(i)$. If this were to occur, we would have $\underline{B}(x) \in \Delta \underline{Q}$ or $\underline{B}(x) \in \Delta \underline{Q}(i)$ for one of the i . It follows that y is joined to the the boundary of $\underline{B}_3(x)$ by an open internal path.

On the other hand, thanks to Lemma 6.4, we know $\underline{B}(x)$ contains a vertex G or an endpoint vertex of $\partial_o G$. Thus, $\underline{B}^+(x)$ contains a vertex $z \in G$, and by the hypothesis on the size of G , we have that z is connected to the boundary of $\underline{B}_3(x)$ by an open internal path. But D and G are disjoint, thus the corresponding paths from y and z to the boundary of $\underline{B}_3(x)$ cannot lie in the same open cluster. We conclude that in the second case, $\underline{B}(x)$ is Type-II. \square

We conclude this subsection with the observation that it is rare for a cube to be either Type-I or Type-II when k is large.

Proposition 6.7. *Let $d \geq 2$ and suppose that $p > p_c(d)$. There are positive constants $c_1(p, d), c_2(p, d)$ so that for each k -cube $\underline{B}(x)$,*

$$\mathbb{P}(\underline{B}(x) \text{ is Type-I}) \leq c_1 \exp(-c_2 k) \quad \text{and} \quad \mathbb{P}(\underline{B}(x) \text{ is Type-II}) \leq c_1 \exp(-c_2 k)$$

Proof. This is in part a consequence of Lemma 7.89 in Grimmett [37]. The proof is also given in Section 3 of [67]. \square

We now apply Zhang's construction to the Cheeger optimizers and handle the additional complications arising from the context of our problem.

6.2. Webbing. The first barrier to a direct application of Zhang's construction to each $G_n \in \mathcal{G}_n$ is that these graphs are not necessarily connected. This is a small issue; Corollary A.9 gives a bound on the number of connected components of each G_n with high probability; we may simply apply the construction of Zhang to each connected component of G_n .

Given $G_n \in \mathcal{G}_n$, we will use the cuts produced by Zhang's argument to construct another discrete set F_n containing G_n . From F_n , we will build a polytope P_n as described at the beginning of the section, and the

uniform bound on the perimeters of the P_n will hinge on our ability to control the size of the cuts produced in Zhang's argument.

For this approach to work at all, we must know that we have not added too many vertices in passing from G_n to F_n , as we would like the empirical measures of both discrete sets to become close when n is large. However, the construction of Zhang gives no direct way of controlling how many additional vertices are enclosed by the "tame" cuts produced. Indeed, upon examining Figure 5, one sees that there is potentially quite a bit of room within the k -cube set \underline{A} . Thus we will apply Zhang's construction on each connected component of G_n , but then we will need to apply this construction a second time to each of the "large" components not in G_n which have been trapped by Zhang's cuts, this time using the construction to excise these large components.

The result will be a discrete object whose boundary consists of a potentially large number of connected components. This creates another complication: in order to control the size of Zhang's cuts, we must work with \mathbb{L}^d -connected objects in order for a Peierls argument to go through in Section 6.3. This is where the so-called webbing comes into play, and this is the reason for working in the setting $d \geq 3$. In this regime, it is possible to tie together all cuts produced by Zhang's construction by a collection of edges of size at most surface order. This approach does not work in the case $d = 2$ for the same reason filaments do not exist in $d = 2$ (see Figure 1).

The first order of business is then to generate all necessary cutsets through Zhang's construction. Let us perform the following operation on each $G_n \in \mathcal{G}_n$. We list the connected components of G_n as $G_n^{(1)}, \dots, G_n^{(M)}$. For each connected component $G_n^{(q)}$ of G_n , let ω'_q be the unique configuration obtained from ω by closing each open edge in $\partial^\omega G_n^{(q)}$. Then within the configuration ω'_q , the edge set $\partial_o G_n^{(q)}$ is a closed cutset separating $G_n^{(q)}$ from ∞ , and we may apply Zhang's construction to produce a closed cutset $\Gamma_n^{(q)}$ which also separates $G_n^{(q)}$ from ∞ . Zhang's construction also produces a cube set $\underline{\Gamma}_n^{(q)}$ which by Lemma 6.5 is \mathbb{L}^d -connected, and satisfies $\Gamma_n^{(q)} \subset (\underline{\Gamma}_n^{(q)})^+$. After applying Zhang's construction in ω_1 first, say, we pass back to the original configuration ω and apply it on ω_2 next. We repeat this for each $q \in \{1, \dots, M\}$. We hope that the n subscript does not cause any confusion in light of our notation for $3k$ -cubes; we emphasize that the sets $\underline{\Gamma}_n^{(q)}$ are collections of k -cubes.

Let ω' be the configuration in which all edges of $\partial^\omega G_n$ are made closed. For an edge set E , we say that a connected subgraph Λ of \mathbf{C}_∞ (in the configuration ω') is *surrounded* by E if every path from Λ to ∞ must use an edge of E . We only care about the connected components of \mathbf{C}_∞ surrounded by E , not other open clusters. Define

$$(6.10) \quad \epsilon(d) := 1 - \frac{d}{(d-1)^2}$$

and observe that $\epsilon(d)$ is positive for $d \geq 3$. In the configuration ω' in which all edges of $\partial^\omega G_n$ are made closed, the cutsets $\Gamma_n^{(q)}$ may surround other connected components of \mathbf{C}_∞ aside from the $G_n^{(q)}$ themselves. If Λ is such a component, we say that Λ is *large* if $|\Lambda| \geq n^{1-\epsilon(d)}$, and we say that Λ is *small* otherwise. Let us enumerate the large components of \mathbf{C}_∞ which are surrounded by any of the cutsets $\Gamma_n^{(q)}$ as L_1, \dots, L_m . We likewise enumerate the small components of \mathbf{C}_∞ which are surrounded by any of the cutsets $\Gamma_n^{(q)}$ as S_1, \dots, S_t . Note that we have suppressed the dependence of the L_i and the S_j on n, ω and G_n . We now define $F_n \subset \mathbf{C}_\infty$ for each G_n as follows,

$$(6.11) \quad F_n := G_n \cup \left(\bigcup_{j=1}^t S_j \right)$$

and we think of F_n as a reasonable approximate to G_n . We now build an edge set which will act as the boundary of F_n . For a given large component L_i , we let ω'_i denote the configuration in which all open edges of $\partial^\omega L_i$ are made closed. For each L_i , we pass from ω to ω'_i and apply Zhang's construction to L_i within the configuration ω'_i before returning to ω . For each L_i then, we produce a collection of edges $\widehat{\Gamma}_n^{(i)}$ which

is a closed cutset in the configuration ω'_i separating L_i from ∞ . We also produce a k -cube set $\widehat{\Gamma}_n^{(i)}$ such that $\widehat{\Gamma}_n \subset (\widehat{\Gamma}_n^{(i)})^+$. For each $G_n \in \mathcal{G}_n$, define the edge set Γ_n and k -cube set $\underline{\Gamma}_n$ as follows:

$$(6.12) \quad \Gamma_n := \left(\bigcup_{q=1}^M \Gamma_n^{(q)} \right) \cup \left(\bigcup_{i=1}^m \widehat{\Gamma}_n^{(i)} \right) \quad \underline{\Gamma}_n := \left(\bigcup_{q=1}^M \underline{\Gamma}_n^{(q)} \right) \cup \left(\bigcup_{i=1}^m \underline{\widehat{\Gamma}}_n^{(i)} \right)$$

We now create a web of paths between each of these cutsets and show that this webbing does not use too many edges.

By fixing an ordering of finite subsets of \mathbb{Z}^d , we may choose in a unique way the endpoint vertex ζ_q of an edge in $\Gamma_n^{(q)}$. We do the same for each $\widehat{\Gamma}_n^{(i)}$, calling these vertices z_i . Thus, once our original configuration ω is set, the optimizer G_n is determined by ω , and through Zhang's construction, so too are the ζ_q and the z_i .

Let $\alpha > 0$ be a parameter to be chosen later. Consider all cubes of the form $[n^\alpha]x + [-[n^\alpha], [n^\alpha]]^d$ which intersect $[-2n, 2n]^d$, where $x \in \mathbb{Z}^d$. Let us list these cubes as $\{\bar{B}_j\}_{j=1}^\ell$. We may assume that the \bar{B}_j are ordered so that consecutive cubes share a face. For n sufficiently large (depending on k), all vertices ζ_q and z_i will lie in the union of the \bar{B}_j . For each $j \in \{1, \dots, \ell\}$, let m_j denote the number of vertices z_i which are contained in the cube \bar{B}_j .

Within each \bar{B}_j , we perform the following procedure: begin with $z_i \in \bar{B}_j$ which is least in our ordering of finite subsets of \mathbb{Z}^d . Pick a path within \mathbb{Z}^d joining this "smallest" vertex to the next smallest z_j within \bar{B}_j . We arrange that this path uses the fewest possible number of edges, and we choose this path in a unique way also.

We continue building unique paths between the current vertex in \bar{B}_j and the next smallest within \bar{B}_j until all vertices z_i within \bar{B}_j have been used. We refer to the union of all paths created in this process as a *tangle* and denote it T_j , which we view as a graph. We repeat this process for each $[n^\alpha]$ -cube \bar{B}_j , defining the tangle T_j to be empty if $m_j = 0$.

We will connect each tangle through a single long path. We construct this path by beginning with \bar{B}_1 and selecting the vertex $z_i \in \bar{B}_1$ minimal in our ordering. We select a minimal vertex from \bar{B}_2 and connect the two successive vertices by a uniquely chosen shortest path in \mathbb{Z}^d , or we do nothing in the case that these two vertices are identical. We repeat this for all consecutive cubes in $\{\bar{B}_j\}_{j=1}^\ell$. If at any point in our process, we find that a cube \bar{B}_j contains no vertices z_i , we take instead the vertex of \mathbb{Z}^d within \bar{B}_j which is minimal in our ordering. When the minimal vertex of the last $[n^\alpha]$ -cube has been used, we link this vertex to the vertex ζ_1 via a uniquely chosen shortest path, and proceed to link successive ζ_q 's by uniquely chosen shortest paths until we reach ζ_M . The union of all paths created in this process shall be denoted String. Note that we have suppressed the n , ω and G_n dependence of both String and the tangles T_j . Let us define

$$(6.13) \quad \text{Web}_n := \text{String} \cup \left(\bigcup_{j=1}^\ell T_j \right)$$

where we now emphasize that $\text{Web}_n = \text{Web}_n(\omega)$ and of course Web_n also depends on the choice of Cheeger optimizer $G_n \in \mathcal{G}_n$. We need to know that with high probability that for each $G_n \in \mathcal{G}_n$, the edge set Web_n has cardinality at most surface order. To show this, we first compute a high probability bound on the number of large components. We use Proposition A.1 and the proof of Corollary A.2 to deduce the following.

Corollary 6.8. *Let $d \geq 3$ and $p > p_c(d)$. There are positive constants $c_1(p, d)$, $c_2(p, d)$ and $c_3(p, d)$ and an almost-surely finite random variable $R' = R'(\omega)$ so that $n \geq R'$ implies that for each ω -connected Λ satisfying $\Lambda \subset C_{2n}$ and $|\Lambda| \geq n^{1-\epsilon(d)}$ we have*

$$|\partial^\omega \Lambda| \geq c_3 |\Lambda|^{(d-1)/d}$$

with the following tail bound on R' :

$$\mathbb{P}_p(R' > n) \leq c_1 \exp(-c_2 n^{1/(d-1)})$$

We will make direct use of Corollary 6.8 in the proof of the following lemma, which is why we have written it here despite it being nearly identical to Corollary A.2.

Lemma 6.9. *Let $d \geq 3$ and $p > p_c(d)$. For each $G_n \in \mathcal{G}_n$, let $m(G_n) = m(G_n, n, \omega)$ denote the number of large components surrounded by the cutsets $\Gamma_n^{(q)}$ in Zhang's construction applied to G_n . Let $M_n = M_n(\omega)$ be the maximum $m(G_n)$ over all $G_n \in \mathcal{G}_n$. There exist positive constants $c_1(p, d, k)$, $c_2(p, d, k)$ and $c_3(p, d)$ so that*

$$\mathbb{P}_p(M_n > c_3 n^{d-1-1/(d-1)}) \leq c_1 \exp(-c_2 n^{1/(d-1)})$$

Proof. Fix $G_n \in \mathcal{G}_n$, let $m = m(G_n)$ and let us work within the high probability event $\{R' \leq n\}$ from Corollary 6.8. By Lemma 6.4, we have that all large components L_i corresponding to G_n are contained in C_{2n} . Thus, on the event $\{R' \leq n\}$, we have for each $i \in \{1, \dots, m\}$:

$$(6.14) \quad |\partial^\omega L_i| \geq cn^{1/(d-1)}$$

using the definition of a large component. If we also work within the high probability event that for all $G_n \in \mathcal{G}_n$, we have $\partial^\omega G_n \leq \eta_3 n^{d-1}$ (from Lemma A.10), we can use this bound in conjunction with (6.14) and the fact that distinct large components must have disjoint open edge boundaries to obtain

$$(6.15) \quad m \leq \frac{\eta_3}{c} n^{d-1-1/(d-1)}$$

on the intersection of the two high probability events invoked. We use the estimates from Lemma A.10 and Corollary 6.8 to complete the proof. \square

With control on the number of large components, we may now compute a high-probability bound on the number of edges in the graph Web_n across all $G_n \in \mathcal{G}_n$. In the following proof, we finally fix the parameter α which controls the dimensions of the cubes \bar{B}_j .

Proposition 6.10. *Let $d \geq 3$ and $p > p_c(d)$. For each $G_n \in \mathcal{G}_n$, we construct the graph $\text{Web}_n = \text{Web}_n(G_n, \omega)$ as above. Let $W_n = W_n(\omega)$ denote the maximum cardinality of $E(\text{Web}_n)$ taken over all Cheeger optimizers. There are positive constants $c_1(p, d, k)$, $c_2(p, d, k)$ and $c_3(p, d)$ so that*

$$\mathbb{P}_p(W_n > c_3 n^{d-1}) \leq c_1 \exp(-c_2 n^{1/(d-1)})$$

Proof. We work within the high probability event from Lemma 6.9 that the maximal number M of large components L_i across all $G_n \in \mathcal{G}_n$ is at most $cn^{d-1-1/(d-1)}$. We also work within the high probability event from Corollary A.9 that the number of connected components of any $G_n \in \mathcal{G}_n$ is at most η_4 .

Let us fix $G_n \in \mathcal{G}_n$ to work with for the rest of the proof. Consider the tangle T_j for G_n associated to the $\lceil n^\alpha \rceil$ -cube \bar{B}_j . Based on our construction of each tangle, the number of edges $|E(T_j)|$ is at most the ℓ^1 -diameter of \bar{B}_j times the number of z_i within \bar{B}_j . Thus,

$$(6.16) \quad \sum_{j=1}^{\ell} |E(T_j)| \leq 8dn^\alpha \sum_{j=1}^{\ell} m_j$$

$$(6.17) \quad \leq c(d)mn^\alpha$$

$$(6.18) \quad \leq c(p, d)n^{\alpha+n-1-1/(d-1)}$$

where in the second line, we have used that each vertex z_i is contained in at most $2d$ distinct $\lceil n^\alpha \rceil$ -cubes, and in the third line we have used our bound from Lemma 6.9. It remains to bound the size of the edge set of String. A shortest \mathbb{Z}^d -path between the vertices of two adjacent $\lceil n^\alpha \rceil$ -cubes uses at most $16n^\alpha$ edges, and

there are at most $c(d)n^{d(1-\alpha)}$ such cubes in total. The final paths in the construction of String which join the vertices ζ_q each use at most $c(d)n$ edges. Thus,

$$(6.19) \quad |E(\text{String})| \leq c(d)n^\alpha n^{d(1-\alpha)} + \eta_4 c(d)n$$

so that upon choosing $\alpha = 1/(d-1)$, we have

$$(6.20) \quad |E(\text{Web}_n)| \leq c(p, d) \left[n^{\alpha+(d-1)-1/(d-1)} + n^{\alpha+d(1-\alpha)} + n \right]$$

$$(6.21) \quad \leq c(p, d)n^{d-1}$$

We use the estimates from Lemma 6.9 and from Corollary A.9 to complete the proof. \square

We are finished working with $[n^\alpha]$ -cubes. Define the coarse-grained image of each Web_n as follows

$$(6.22) \quad \underline{\text{Web}}_n := \{\underline{B}(x) : \underline{B}(x) \cap \text{Web}_n \neq \emptyset\}$$

so that each $\underline{\text{Web}}_n$ is a collection of k -cubes depending on n, ω and G_n . Regardless of the size of each $E(\text{Web}_n)$, the construction of $\underline{\text{Web}}_n$ along with Lemma 6.5 yield the following.

Lemma 6.11. *For each $G_n \in \mathcal{G}_n$, the k -cube set $\underline{\Gamma}_n \cup \underline{\text{Web}}_n$ corresponding to G_n is \mathbb{L}^d -connected.*

This is all the information we need about the webbing.

6.3. A Peierls argument. Using the webbing in conjunction with Zhang's construction, we can show that when k is chosen sufficiently large, the size of each Γ_n is with high probability surface order. Let us make a small observation before diving into the Peierls argument.

Lemma 6.12. *For each $G_n \in \mathcal{G}_n$, any edge of \mathbb{Z}^d is contained in at most $(11k)^d$ distinct edge sets among the $\Gamma_n^{(q)}$ and the $\widehat{\Gamma}_n^{(i)}$ corresponding to G_n .*

Proof. Fix $G_n \in \mathcal{G}_n$. If $\Gamma_n^{(q)}$ uses an edge $e \in E(\mathbb{Z}^d)$, there is a k -cube $\underline{B}(x) \in \underline{\Gamma}_n^{(q)}$ so that $e \in \underline{B}^+(x)$. By Lemma 6.4, \underline{B}^+ also contains a vertex $y \in G_n^{(q)}$. On the other hand, suppose another cutset, say $\widehat{\Gamma}_n^{(i)}$ uses the same edge e . Identical reasoning tells us that the five-fold dilate of $\underline{B}(x)$

$$(6.23) \quad \underline{B}_5(x) := 2kx + [-5k, 5k]^d$$

contains both the vertex y and a vertex $z \in L_i$. If other cutsets corresponding to other connected components of G_n or other large L_i also use e , at least one vertex of each of these graphs must also lie within $\underline{B}_5(x)$. As the components of G_n and the L_i are disjoint, and because $\underline{B}_5(x)$ contains at most $(11k)^d$ vertices, the desired claim holds. \square

We now proceed with the Peierls argument, in which the renormalization parameter k is chosen once and for all.

Proposition 6.13. *Let $d \geq 3$ and $p > p_c(d)$. There exists $\gamma = \gamma(p, d)$ and positive constants $c_1(p, d)$ and $c_2(p, d)$ so that*

$$\mathbb{P}_p \left(\max_{G_n \in \mathcal{G}_n} |\Gamma_n| \geq \gamma n^{d-1} \right) \leq c_1 \exp(-c_2 n^{1/(d-1)})$$

Proof. Let \mathcal{E}_{web} be the event from Proposition 6.10 that for all $G_n \in \mathcal{G}_n$, the corresponding graphs Web_n satisfy $|E(\text{Web}_n)| \leq c(p, d)n^{d-1}$. We work with a fixed $G_n \in \mathcal{G}_n$ and corresponding Γ_n throughout the proof, and first use the bounds in Proposition 6.10 and in Lemma A.10 as follows:

$$(6.24) \quad \mathbb{P}_p(|\Gamma_n| \geq \gamma n^{d-1}) \leq c_1 \exp(-c_2 n^{1/(d-1)}) + \mathbb{P}_p(\{|\Gamma_n| \geq \gamma n^{d-1}\} \cap \{|\partial^\omega G_n| \leq \eta_3 n^{d-1}\} \cap \mathcal{E}_{\text{web}})$$

$$(6.25) \quad \leq c_1 \exp(-c_2 n^{1/(d-1)}) + \sum_{j=\gamma n^{d-1}}^{\infty} \mathbb{P}_p(\{|\Gamma_n| = j\} \cap \{|\partial^\omega G_n| \leq (\eta_3/\gamma)j\} \cap \mathcal{E}_{\text{web}})$$

Take γ large depending on k, p and d so that $\eta_3/\gamma < [2 \cdot 4^d(11k)^d(4k)^{d+1}]^{-1}$.

$$(6.26) \quad \mathbb{P}_p(|\Gamma_n| \geq \gamma n^{d-1}) \leq c_1 \exp(-c_2 n^{1/(d-1)})$$

$$(6.27) \quad + \sum_{j=\gamma n^{d-1}}^{\infty} \mathbb{P}_p(|\Gamma_n| = j) \cap \{|\partial^\omega G_n| < [2 \cdot 4^d(11k)^d(4k)^{d+1}]^{-1} j\} \cap \mathcal{E}_{\text{web}}$$

We equip $\underline{\Gamma}_n$ with a graph structure: the vertices of this graph will be the collection of k -cubes in $\underline{\Gamma}_n$, and we create an edge between two vertices $\underline{B}(x)$ and $\underline{B}(y)$ if $x \sim_{\mathbb{L}} y$. The maximum degree of any vertex in this graph is 3^d , so by Turán's theorem (Theorem A.11 in the appendix), there exists a subcollection of cubes $\underline{\Gamma}'_n \subset \underline{\Gamma}_n$ so that whenever $\underline{B}(x), \underline{B}(y) \in \underline{\Gamma}'_n$, the corresponding $3k$ -cubes $\underline{B}_3(x)$ and $\underline{B}_3(y)$ have disjoint interiors. Moreover, Turán gives that $|\underline{\Gamma}'_n| \geq |\underline{\Gamma}_n|/4^d$. Thus,

$$(6.28) \quad |\underline{\Gamma}'_n| \geq \frac{|\underline{\Gamma}_n|}{4^d} \geq \frac{|\underline{\Gamma}_n|}{4^d(11k)^d(4k)^{d+1}}$$

as when $k \geq d$, there are at most $(4k)^{d+1}$ edges of \mathbb{Z}^d which have an endpoint in a given augmented k -cube, and thanks to Lemma 6.12, we know each such edge is contained in at most $(11k)^d$ distinct cutsets among the $\widehat{\Gamma}_n^{(q)}$ and $\widehat{\Gamma}_n^{(i)}$. Now consider the following event:

$$(6.29) \quad \{|\Gamma_n| = j\} \cap \{|\partial^\omega G_n| < [2 \cdot 4^d(11k)^d(4k)^{d+1}]^{-1} j\} \cap \mathcal{E}_{\text{web}}$$

Within the event given in (6.29), we know from (6.28) that at most half of the cubes of $\underline{\Gamma}'_n$ can contain an edge of $\partial^\omega G_n$. Thus there is a further subcollection $\underline{\Gamma}''_n \subset \underline{\Gamma}'_n$ so that

$$(6.30) \quad |\underline{\Gamma}''_n| \geq \frac{|\underline{\Gamma}'_n|}{2 \cdot 4^d(11k)^d(4k)^{d+1}}$$

and such that each k -cube of $\underline{\Gamma}''_n$ is either Type-I or Type-II by Proposition 6.6.

Of course, $\underline{\Gamma}''_n$ inherits from $\underline{\Gamma}'_n$ the property that any two $\underline{B}(x), \underline{B}(y) \in \underline{\Gamma}''_n$ are such that $\underline{B}_3(x)$ and $\underline{B}_3(y)$ have disjoint interiors. Thus, for distinct $\underline{B}(x)$ and $\underline{B}(y)$ in $\underline{\Gamma}''_n$, the event that $\underline{B}(x)$ is Type-I or Type-II is independent from the event that $\underline{B}(y)$ is Type-I or Type-II.

We continue to work within the event given in (6.29). Write $s = |\underline{\Gamma}''_n|$; on the event \mathcal{E}_{web} , we have also that $|\text{Web}_n| \leq c(p, d)n^{d-1}$, where we perhaps alter c in a way depending on d . By Proposition 6.11, the k -cube set $\underline{\Gamma}_n \cup \text{Web}_n$ is \mathbb{L}^d -connected, so we use Proposition A.12 to deduce that there are at most

$$(6.31) \quad (3n)^d [c(d)]^{s+cn^{d-1}}$$

ways to choose the k -cube set $\underline{\Gamma}_n \cup \text{Web}_n$. The factor of $(3n)^d$ is a crude upper bound on the number of vertices of \mathbb{Z}^d within $[-n, n]^d$. There are at most $2^{s+cn^{d-1}}$ ways to choose $\underline{\Gamma}_n$ from $\underline{\Gamma}_n \cup \text{Web}_n$, at most $2^{s+cn^{d-1}}$ ways to choose $\underline{\Gamma}'_n$ from $\underline{\Gamma}_n$ and at most $2^{s+cn^{d-1}}$ ways to choose $\underline{\Gamma}''_n$ from $\underline{\Gamma}'_n$. A union bound then gives

$$(6.32) \quad \mathbb{P}_p(|\Gamma_n| = j) \cap \{|\partial^\omega G_n| < [2 \cdot 4^d(11k)^d(4k)^{d+1}]^{-1} j\} \cap \mathcal{E}_{\text{web}} \leq (3n)^d [c(d)]^{s+cn^{d-1}} [c_1 \exp(-c_2 k)]^{s/(2 \cdot 4^d)}$$

Here, we've used the lower bound on $|\underline{\Gamma}''_n|$ in terms of s following from (6.28), as well as the bounds of Proposition 6.7 and the independence of the events to which these bounds are applied. Note that from (6.28) we have $s \geq j/(11k)^d(4k)^{d+1}$, and as $j \geq \gamma n^{d-1}$, we may take γ larger if necessary, again in a way depending on k, p and d , so that $s \geq cn^{d-1}$, giving

$$(6.33) \quad \mathbb{P}_p(|\Gamma_n| = j) \cap \{|\partial^\omega G_n| < [2 \cdot 4^d(11k)^d(4k)^{d+1}]^{-1} j\} \cap \mathcal{E} \leq (3n)^d [c(d)]^{2s} [c_1 \exp(-c_2 k)]^{s/(2 \cdot 4^d)}$$

We now choose k large enough in a way depending on p and d so that $[c(d)]^2 [c_1 \exp(-c_2 k)]^{1/2 \cdot 4^d} < e^{-1}$, at which point we consider k fixed. For this choice of k ,

$$(6.34) \quad \mathbb{P}_p(|\Gamma_n| = j) \cap \{|\partial^\omega G_n| < [2 \cdot 4^d \eta_4(4k)^{d+1}]^{-1} j\} \cap \mathcal{E} \leq (3n)^d \exp(-s)$$

and we use this bound in conjunction with (6.26) and (6.28) to obtain

$$(6.35) \quad \mathbb{P}_p(|\Gamma_n| \geq \gamma n^{d-1}) \leq c_1 \exp(-c_2 n^{1/(d-1)}) + \sum_{j=\gamma n^{d-1}}^{\infty} (3n)^d \exp(-s)$$

$$(6.36) \quad \leq c_1 \exp(-c_2 n^{1/(d-1)}) + \sum_{j=\gamma n^{d-1}}^{\infty} (3n)^d \exp\left(-j/[(11k)^d(4k)^{d+1}]\right)$$

We choose γ sufficiently large depending on p, d and $k(p, d)$ to complete the proof. \square

We emphasize that the renormalization parameter $k = k(p, d)$ has now been fixed. Recall that each F_n is the union of G_n with the small components S_j corresponding to G_n . We asserted that the associated edge set Γ_n could be thought of as the boundary of F_n ; the following proposition makes this rigorous. Note that, as a result of Lemma 6.4, we can only conclude that each $F_n \subset [-n - 2k, n + 2k]^d$ as opposed to $F_n \subset [-n, n]^d$.

Proposition 6.14. *Let $d \geq 3$ and $p > p_c(d)$. Define $\ell(n) := \lfloor n^{(1-\epsilon(d))/2d} \rfloor$, and let B denote a closed axis-parallel cube of side-length $2\ell(n)$ centered at an element of \mathbb{Z}^d with $B \cap [-n - 2k, n + 2k]^d$ non-empty. There exist positive constants $c_1(p, d)$ and $c_2(p, d)$ so that with probability at least*

$$1 - c_1 \exp\left(-c_2 n^{(1-\epsilon(d))/2d}\right)$$

whenever F_n corresponds to any $G_n \in \mathcal{G}_n$, and whenever $B \cap F_n \neq \emptyset$ and $B \cap F_n \neq B \cap C_\infty$ for some B as above, we have that B either contains an endpoint vertex of an edge in Γ_n , or that the three-fold dilate of B around its center contains an endpoint vertex of an edge in $\partial^\omega G_n$.

Proof. Say that $B = [-\ell(n), \ell(n)]^d + x$ for some $x \in \mathbb{Z}^d$, with $B \cap [-n - 2k, n + 2k]^d \neq \emptyset$. We define augmented and dilated versions of B as in the case of k -cubes:

$$(6.37) \quad B^+ := [-\ell(n) - 1, \ell(n) + 1]^d + x$$

$$(6.38) \quad B_3 := [-3\ell(n), 3\ell(n)]^d + x$$

Suppose that $B \cap F_n \neq \emptyset$ and that $B \cap F_n \neq B \cap C_\infty$. Suppose first that B contains $y \in C_\infty \setminus F_n$ which is connected to ∞ by a (not necessarily open) path γ' which uses no edges of Γ_n . As $B \cap F_n \neq \emptyset$, B contains some vertex z which is either a member of some $G_n^{(q)}$ or some small component S_j . If γ is any path from z to y within B , it must be that γ uses an edge of Γ_n , otherwise we could concatenate γ with γ' to show that either some $G_n^{(q)}$ or some S_j is not surrounded by the edge set Γ_n .

Thus we may suppose every vertex $y \in (C_\infty \setminus F_n) \cap B$ is surrounded by one of the cutsets $\Gamma_n^{(q)}$ or $\widetilde{\Gamma}_n^{(i)}$. Any y with this property must lie in some large component L_i . Let us again choose some $z \in F_n \cap B$. Suppose that B_3 contains no endpoint of an edge in $\partial^\omega G_n$. If this is the case, there can be no open path between z and y in B_3 . On the other hand, if we work within the high probability event from Lemma A.8 that for each $G_n \in \mathcal{G}_n$, every connected component of G_n satisfies $|G_n^{(q)}| \geq \eta_1 n^d$, we may conclude that for all n sufficiently large, we cannot have the connected component of F_n which contains z contained itself in B_3 . Likewise, by the largeness of each L_i , and due to our choice of $\ell(n)$, it cannot be that L_i is contained in B_3 .

Thus there is an open path from z to $\partial(B_3 \cap \mathbb{Z}^d)$ as well as an open path from y to $\partial(B_3 \cap \mathbb{Z}^d)$, and these paths are not joined by any open path within B_3 . We have shown that, within this second scenario that B does not contain an edge of Γ_n , if B_3 contains no endpoint of an edge in $\partial^\omega G_n$ then B_3 has the Type-II property. Proposition 6.7 in conjunction with a union bound tells us that the existence of a cube B of side-length $2\ell(n)$ with the Type-II property is at most

$$(6.39) \quad (3n)^d c_1 \exp(-c_2 n^{(1-\epsilon(n))/2d})$$

so that working within the compliment of this event and the aforementioned high probability event from Lemma A.8, we have the desired result. \square

Recall that in Section 2, from each $G_n \in \mathcal{G}_n$ we formed the empirical measure $\mu_n \in \mathcal{M}([-1, 1]^d)$ associated to G_n . For each F_n built from a given $G_n \in \mathcal{G}_n$, we define the *empirical measure* $\tilde{\mu}_n$ associated to F_n similarly as

$$(6.40) \quad \tilde{\mu}_n := \frac{1}{n^d} \sum_{x \in F_n} \delta_{x/n}$$

Note that $\tilde{\mu}_n \in \mathcal{M}([-1 - 2k/n, 1 + 2k/n]^d)$. We close Section 6 by observing that μ_n and $\tilde{\mu}_n$ roughly agree on Borel sets.

Lemma 6.15. *Let $d \geq 3$ and let $p > p_c(d)$. There exist positive constants $c_1(p, d)$, $c_2(p, d)$ and $\eta_3(p, d)$ so that for each Borel $K \subset [-1 - 2k/n, 1 + 2k/n]^d$,*

$$\mathbb{P} \left(\max_{G_n \in \mathcal{G}_n} |\mu_n(K) - \tilde{\mu}_n(K)| > \eta_3 n^{-\epsilon(d)} \right) \leq c_1 \exp(-c_2 n^{(d-1)/2d})$$

Proof. Let us work within the high probability event from Lemma A.10 that for all $G_n \in \mathcal{G}_n$, we have $|\partial^\omega G_n| \leq \eta_3 n^{d-1}$. For each small component S_j , the open edge boundary $\partial^\omega S_j$ contains at least one edge in $\partial^\omega G_n$. The edge sets $\{\partial^\omega S_j\}_{j=1}^t$ are pairwise disjoint, thus the number t of small components S_j is at most $\eta_3 n^{d-1}$. From the definition of a small component, we observe that for each $G_n \in \mathcal{G}_n$,

$$(6.41) \quad |F_n \setminus G_n| = \sum_{j=1}^t |S_j| \leq \eta_3 n^{d-\epsilon(d)}$$

The claim follows from the definitions of empirical measures for G_n and F_n . \square

In the following section, we “flatten” each $\tilde{\mu}_n$ into a measure representing a set of finite perimeter.

7. CONTIGUITY

This section is the second and final step in a procedure taking us from discrete objects to continuous objects. The constants k and γ from Proposition 6.13 have now been fixed. We will no longer use k -cubes in this section, and the constant k will not come up except to say that the empirical measures $\tilde{\mu}_n$ are elements of $\mathcal{M}([-1 - 2k/n, 1 + 2k/n]^d)$.

We will use another renormalization argument in this section, this time at a different scale. It is convenient to use the same notation. Let $\ell(n) = \lfloor n^{(1-\epsilon(d))/2d} \rfloor$, and suppress the dependence of on n by writing $\ell(n)$ as ℓ . We *redefine* $\underline{B}(x)$ to be the ℓ -cube $(2\ell)x + [-\ell, \ell]^d$. We deal briefly with 3ℓ -cubes, and we also need to introduce the notion of a *reduced* ℓ -cube. The reduced ℓ -cube associated to x is

$$(7.1) \quad \underline{B}^-(x) := (2\ell)x + [-(\ell - 1), \ell - 1]^d$$

Let us fix a $G_n \in \mathcal{G}_n$. Given a finite collection of edges E , we let $\text{Hull}(E)$ denote the collection of vertices $x \in \mathbb{Z}^d$ such that any path from x to ∞ must use an edge of E . Recall that for any $x \in \mathbb{Z}^d$, we defined the dual cube $Q(x)$ as $[-1/2, 1/2]^d + x$. Let A_q index the large components L_i associated to G_n which are surrounded by $\Gamma_n^{(q)}$. For the fixed G_n , let $H_n^{(q)}$ denote

$$(7.2) \quad H_n^{(q)} := \text{Hull}(\Gamma_n^{(q)}) \setminus \left(\bigcup_{i \in A_q} \text{Hull}(\tilde{\Gamma}_n^{(i)}) \right)$$

and let $H_n = \bigcup_{q=1}^M H_n^{(q)}$. Define the polytope

$$(7.3) \quad P_n = \left(\bigcup_{x \in H_n} Q(x) \right) \cap [-n, n]^d$$

We define $\nu_n = \nu_n(\omega, G_n)$ to be the representative measure of $(1/n)P_n$ (see Section 2), so that $\nu_n \in \mathcal{M}([-1, 1]^d)$. We repeat this construction for each $G_n \in \mathcal{G}_n$, and we will show that for each G_n , the measures $\mu_n, \tilde{\mu}_n$ and ν_n are all \mathfrak{d} -close. Before doing so, we must develop our understanding of each H_n .

Each large component L_i associated to G_n is surrounded by some cutset $\Gamma_n^{(q)}$, also associated to G_n . We say that a large component L_i is *bad* if L_i is surrounded by $\Gamma_n^{(q)}$ and the corresponding connected component $G_n^{(q)}$ of G_n is surrounded by the cutset $\widehat{\Gamma}_n^{(i)}$ corresponding to L_i . Such components are bad because if they exist, subtracting the hull of $\widehat{\Gamma}_n^{(i)}$ associated to these components from the hull of $\Gamma_n^{(q)}$ removes $G_n^{(q)}$. If such components exist, we cannot expect ν_n and μ_n to be \mathfrak{d} -close. Fortunately, the following lemma tells us these bad components do not exist.

Lemma 7.1. *For each $G_n \in \mathcal{G}_n$, consider the large components L_i associated to G_n , and let $G_n^{(q)}$ be a connected component of G_n . It is impossible for any large component L_i to be bad: that is, L_i cannot be surrounded by some $\Gamma_n^{(q)}$ associated to $G_n^{(q)}$, with this $G_n^{(q)}$ itself surrounded by $\widehat{\Gamma}_n^{(i)}$.*

Proof. Fix $G_n \in \mathcal{G}_n$. Suppose that for a large component L_i and a connected component $G_n^{(q)}$ of the G_n , we have both that L_i is surrounded by $\Gamma_n^{(q)}$ and that $G_n^{(q)}$ is surrounded by $\widehat{\Gamma}_n^{(i)}$.

Recall that ω_q is the configuration obtained from ω by closing each open edge in $\partial^\omega G_n^{(q)}$. Because L_i is surrounded by $\Gamma_n^{(q)}$, it must be that $\partial_o L_i$ is a closed cutset separating L_i from ∞ in the configuration ω_q . Thus, back in the configuration ω , it follows that $\partial_o L_i \cap \partial^\omega L_i \subset \partial^\omega G_n^{(q)}$.

Let us see how this gives rise to a contradiction: let $y \in G_n^{(q)}$. Working in the original configuration ω , consider a simple path γ from y to ∞ within \mathbf{C}_∞ , so that γ uses only open edges. We may assume that y is the only vertex of $G_n^{(q)}$ used by γ , as γ must eventually leave $G_n^{(q)}$ and not return. Because $G_n^{(q)}$ is surrounded by $\widehat{\Gamma}_n^{(i)}$, it must be that γ uses an open edge e in $\widehat{\Gamma}_n^{(i)}$. As $\widehat{\Gamma}_n^{(i)}$ is a closed cutset in the configuration ω'_i , this open edge e must lie in $\partial^\omega L_i$. Thus we have shown γ uses a vertex of L_i , and thus γ contains an open path from L_i to ∞ which uses no vertices of $G_n^{(q)}$. This is impossible, as $\partial_o L_i \cap \partial^\omega L_i \subset \partial^\omega G_n^{(q)}$. \square

Let us extract a useful observation from the proof of Lemma 7.1.

Lemma 7.2. *Let $G_n \in \mathcal{G}_n$. Each large component L_i corresponding to G_n is surrounded by exactly one cutset $\Gamma_n^{(q)}$ corresponding to a connected component of G_n .*

Proof. We work with a fixed $G_n \in \mathcal{G}_n$. Each large component L_i associated to G_n is surrounded by at least one of the cutsets $\Gamma_n^{(q)}$. Suppose that L_i is surrounded by both $\Gamma_n^{(q)}$ and by $\Gamma_n^{(q')}$ for $q \neq q'$. Thanks to the proof of Lemma 7.1, we know $\partial_o L_i \cap \partial^\omega L_i \subset \partial^\omega G_n^{(q)}$ and $\partial_o L_i \cap \partial^\omega L_i \subset \partial^\omega G_n^{(q')}$. As $L_i \subset \mathbf{C}_\infty$, the edge set $\partial_o L_i \cap \partial^\omega L_i$ is non-empty, thus $\partial^\omega G_n^{(q)} \cap \partial^\omega G_n^{(q')}$ is non-empty. But this is impossible, as distinct connected components of G_n must have disjoint open edge boundary. \square

Using Lemma 7.1, we show that each F_n is contained in the corresponding vertex set H_n .

Lemma 7.3. *For each $G_n \in \mathcal{G}_n$, we have the containment $F_n \subset H_n$.*

Proof. Fix $G_n \in \mathcal{G}_n$ as usual. We begin the following claim: if $y \in F_n$ and $y \notin H_n$, then there exists a large component L_i and a connected component $G_n^{(q)}$ of G_n such that y is surrounded by both $\Gamma_n^{(q)}$ and $\widehat{\Gamma}_n^{(i)}$, and such that L_i is itself surrounded by $\Gamma_n^{(q)}$.

This claim follows directly from the definition of H_n : as $y \in F_n$, we have $y \in \text{Hull}(\Gamma_n^{(q)})$ for some q . The only way we could have $y \notin H_n$ is if y were surrounded by some $\widehat{\Gamma}_n^{(i)}$ for $i \in A_q$, and we recall that A_q indexes the large components L_i which are surrounded by $\Gamma_n^{(q)}$.

With this initial claim settled, we now consider the large component L_i given by the claim and we pass to the configuration ω'_i in which each edge of $\partial^\omega L_i$ made closed. In this configuration, $\widehat{\Gamma}_n^{(i)}$ consists only

of closed edges. Let Λ be the open cluster containing y in this configuration, so that by hypothesis Λ is surrounded by $\widehat{\Gamma}_n^{(i)}$.

Let $F_n^{(q)}$ denote the connected component of F_n containing $G_n^{(q)}$, and let $z \in F_n^{(q)}$. Suppose for the sake of contradiction that $z \notin \Lambda$. Let γ be a path from y to z within $F_n^{(q)}$, so that in the original configuration ω , the path γ uses only open edges. From the assumption $z \notin \Lambda$, if we pass to the configuration ω_i , we see that γ must use a closed edge e , which is necessarily an element of $\partial^\omega L_i$ back in the configuration ω .

Because γ is a path in $F_n^{(q)}$, it must join two vertices which are either in G_n or in one of the small components S_j . But $e \in \partial^\omega L_i$, which means an endpoint of e must also lie in L_i . It is impossible for e to satisfy all these requirements. Thus, we conclude that $F_n^{(q)} \subset \Lambda$, and consequently $G_n^{(q)} \subset \Lambda$. In particular, this means $G_n^{(q)}$ is surrounded by $\widehat{\Gamma}_n^{(i)}$, which implies (through the very first claim in this proof) that L_i is a bad component. We apply Lemma 7.1 to complete the proof. \square

We finally use Proposition 6.13 to control the perimeter of each P_n .

Corollary 7.4. *Let $d \geq 3$ and $p > p_c(d)$. There are positive constants $c_1(p, d)$, $c_2(p, d)$ and $\gamma(p, d)$ so that*

$$\mathbb{P}_p \left(\max_{G_n \in \mathcal{G}_n} \text{per}(P_n) \geq \gamma n^{d-1} \right) \leq c_1 \exp(-c_2 n^{1/(d-1)})$$

Proof. We work within the high probability event \mathcal{E} corresponding to Proposition 6.13 that for each $G_n \in \mathcal{G}_n$, the corresponding cutset Γ_n satisfies $|\Gamma_n| < \gamma n^{d-1}$. Define the polytope \tilde{P}_n as follows.

$$(7.4) \quad \tilde{P}_n := \bigcup_{x \in H_n} Q(x)$$

Every boundary face of \tilde{P} has \mathcal{H}^{d-1} -measure one, and each boundary face of \tilde{P} corresponds to a unique edge of Γ_n , thus within the event \mathcal{E} , we see \tilde{P}_n has perimeter at most γn^{d-1} . As $P_n = \tilde{P}_n \cap [-n, n]^d$, we observe that the perimeter of P_n is at most $\gamma + 2d(2n)^{d-1}$, which completes the proof upon redefining γ . \square

We now proceed to the contiguity argument which will show μ_n and ν_n are \mathfrak{d} -close with high probability.

7.1. A contiguity argument. We now adapt the argument given in Section 16.2 of [18]. Let $\epsilon > 0$, and introduce the \mathbb{Z}^d -process $\{Z_x^{(\delta)}\}_{x \in \mathbb{Z}^d}$, where each $Z_x^{(\delta)}$ shall be the indicator function of the event

$$\left\{ \frac{|\mathbf{C}_\infty \cap \underline{B}^-(x)|}{\mathcal{L}^d(\underline{B}^-(x))} \in (\theta_p(d) - \delta, \theta_p(d) + \delta) \right\}$$

These random variables are independent because they are defined in terms of reduced ℓ -cubes. This independence allows us to avoid using, for instance, the powerful machinery of Liggett, Schonmann and Stacey [44]. Using Corollary A.4 on each ℓ -cube intersecting $[-n, n]^d$, we will show $\tilde{\mu}_n$ and ν_n are \mathfrak{d} -close.

Proposition 7.5. *Let $d \geq 3$ and let $p > p_c(d)$. Let $Q \subset [-1, 1]^d$ be an axis-parallel cube. For all $\delta > 0$, there are positive constants $c_1(p, d, \delta)$ and $c_2(p, d, \delta)$ so that*

$$\mathbb{P}_p \left(\max_{G_n \in \mathcal{G}_n} |\tilde{\mu}_n(Q) - \nu_n(Q)| \geq \delta \right) \leq c_1 \exp(-c_2 n^{(1-\epsilon(d))/2d})$$

Proof. Fix $G_n \in \mathcal{G}_n$, and let F_n , $\tilde{\mu}_n$, P_n and ν_n be the objects constructed above for this G_n . Let \underline{L} denote the following collection of ℓ -cubes:

$$(7.5) \quad \underline{L} = \{\underline{B}(x) : \underline{B}(x) \cap [-n - 2k, n + 2k]^d \neq \emptyset\}$$

and let \underline{L}° denote the collection of ℓ -cubes in \underline{L} which are contained in $[-n, n]^d$. Observe that for each ℓ -cube $\underline{B}(x)$, we have the bounds

$$(7.6) \quad \tilde{\mu}_n(n^{-1} \underline{B}(x)) \leq (3\ell/n)^d \text{ and } \nu_n(n^{-1} \underline{B}(x)) \leq (3\ell/n)^d$$

As $|\underline{L} \setminus \underline{L}^\circ| \leq c(d)n^{d-1}$, it follows that

$$(7.7) \quad |\tilde{\mu}_n(Q) - \nu_n(Q)| \leq c(d) \frac{\ell^d}{n} + \sum_{\underline{B}(x) \in \underline{L}^\circ} |\tilde{\mu}_n(Q \cap n^{-1}\underline{B}(x)) - \nu_n(Q \cap n^{-1}\underline{B}(x))|$$

We now pass to the corresponding reduced ℓ -cubes in \underline{L}° , so that we can exploit the independence of the collection $\{Z_x^{(\delta)}\}_{x \in \mathbb{Z}^d}$ at the correct time. Observe that for each $\underline{B}(x) \in \underline{L}$, we have

$$(7.8) \quad \tilde{\mu}_n(n^{-1}(\underline{B}(x) \setminus \underline{B}^-(x))) \leq c(d)\ell^{d-1}/n^d \quad \text{and} \quad \nu_n(n^{-1}(\underline{B}(x) \setminus \underline{B}^-(x))) \leq c(d)\ell^{d-1}/n^d$$

As there are at most $c(d)n^d/\ell^d$ such ℓ -cubes in \underline{L} , it follows that

$$(7.9) \quad |\tilde{\mu}_n(Q) - \nu_n(Q)| \leq c(d) \frac{\ell^d}{n} + c(d) \frac{1}{\ell} + \sum_{\underline{B}(x) \in \underline{L}^\circ} |\tilde{\mu}_n(Q \cap n^{-1}\underline{B}^-(x)) - \nu_n(Q \cap n^{-1}\underline{B}^-(x))|$$

As Q is an axis-parallel-cube, its boundary intersects at most $c(d)n^{d-1}$ ℓ -cubes. By modifying the constant in front of the ℓ^d/n term above, we may then assume that all ℓ -cubes in \underline{L}' do not intersect the boundary of Q , so that

$$(7.10) \quad |\tilde{\mu}_n(Q) - \nu_n(Q)| \leq c(p, d) \frac{\ell^d}{n} + c(d) \frac{1}{\ell} + \sum_{\underline{B}(x) \in \underline{L}^\circ} |\tilde{\mu}_n(n^{-1}\underline{B}^-(x)) - \nu_n(n^{-1}\underline{B}^-(x))|$$

Define the following high probability events:

$$(7.11) \quad \mathcal{E}_1 := \left\{ \max_{G_n \in \mathcal{G}_n} \text{per}(P_n) < \gamma n^{d-1} \right\} \quad \mathcal{E}_2 := \left\{ \max_{G_n \in \mathcal{G}_n} |\Gamma_n| < \gamma n^{d-1} \right\} \quad \mathcal{E}_3 := \left\{ \max_{G_n \in \mathcal{G}_n} |\partial^\omega G_n| \leq \eta_3 n^{d-1} \right\}$$

so that \mathcal{E}_1 , \mathcal{E}_2 and \mathcal{E}_3 are respectively high probability events from Corollary 7.4, Proposition 6.13 and Lemma A.10. Finally, let \mathcal{E}_4 be the high probability event from Proposition 6.14. We work within the intersection of \mathcal{E}_1 through \mathcal{E}_4 ; these events allow us to treat P_n and F_n as objects whose perimeters are surface order.

Motivated by the event \mathcal{E}_4 , let $\underline{L}' \subset \underline{L}^\circ$ be the collection of ℓ -cubes $\underline{B}(x) \in \underline{L}^\circ$ such that $\underline{B}(x) \cap P_n = \emptyset$ or $\underline{B}(x) \cap P_n = \underline{B}(x)$ and $\underline{B}(x) \cap F_n = \emptyset$ or $\underline{B}(x) \cap F_n = \underline{B}(x) \cap \mathbf{C}_\infty$. We invoke the bounds from our high probability events to obtain the following.

$$(7.12) \quad |\tilde{\mu}_n(Q) - \nu_n(Q)| \leq c(p, d) \frac{\ell^d}{n} + c(d) \frac{1}{\ell} + \sum_{\underline{B}(x) \in \underline{L}'} |\tilde{\mu}_n(n^{-1}\underline{B}^-(x)) - \nu_n(n^{-1}\underline{B}^-(x))|$$

Let \underline{L}'' be the collection of ℓ -cubes in \underline{L}° such that $F_n \cap \underline{B}(x) = \mathbf{C}_\infty \cap \underline{B}(x)$ and $P_n \cap \underline{B}(x) = \underline{B}(x)$ or such that $F_n \cap \underline{B}(x) = \emptyset$ and $P_n \cap \underline{B}(x) = \emptyset$. It is immediate that $\underline{L}'' \subset \underline{L}'$, but we make the even stronger claim that $\underline{L}'' = \underline{L}'$. To prove this, we show that two of the four possible cases for cubes defining \underline{L}' are impossible.

Case I: $F_n \cap \underline{B}(x) = \mathbf{C}_\infty \cap \underline{B}(x)$ and $P_n \cap \underline{B}(x) = \emptyset$. This case is handled entirely by Lemma 7.3: as $F_n \subset H_n$, this is impossible unless $\mathbf{C}_\infty \cap \underline{B}(x) = \emptyset$, which is one of the two options we allow for.

Case II: $F_n \cap \underline{B}(x) = \emptyset$ and $P_n \cap \underline{B}(x) = \underline{B}(x)$. If $\mathbf{C}_\infty \cap \underline{B}(x) = \emptyset$, we are in one of the two allowed options, so we may assume this is not the case. Let $y \in \mathbf{C}_\infty \cap \underline{B}(x)$. As $P_n \cap \underline{B}(x) = \underline{B}(x)$, it follows that $y \in H_n$. Thus y is surrounded by some $\Gamma_n^{(q)}$, so that either $y \in F_n$ or $y \in L_i$ for some i . The former option is impossible by hypothesis, so we may conclude $y \in L_i$ for some i . By Lemma 7.2, this large component L_i containing y is surrounded by exactly one of the $\Gamma_n^{(q)}$, so that $y \in H_n$ implies $y \in H_n^{(q)}$ and $y \notin H_n^{(q')}$ whenever $q' \neq q$. But in the construction of $H_n^{(q)}$, we see that the hull of $\widehat{\Gamma}_n^{(i)}$ is removed from the hull of $\Gamma_n^{(q)}$. As the hull of $\widehat{\Gamma}_n^{(i)}$ contains L_i and hence y , it is also impossible that $y \in H_n^{(q)}$. This is a contradiction.

Having settled both cases, we conclude that $\underline{L}'' = \underline{L}'$. Let us form one last high probability event \mathcal{E}_5 concerning the collection $\{Z_x^{(\delta)}\}_{x \in \mathbb{Z}^d}$: we let \mathcal{E}_5 be the event that $Z_x^{(\delta)} = 1$ for all x with $\underline{B}(x) \in \underline{L}''$. As a

consequence of Corollary A.4,

$$(7.13) \quad \mathbb{P}(\mathcal{E}_5^c) \leq c(d)n^d c_1 \exp(-c_2 n^{(1-\epsilon(d))/2d})$$

where $c_1 = c_1(p, d, \delta)$ and $c_2 = c_2(p, d, \delta)$. Working within the intersection of \mathcal{E}_1 through \mathcal{E}_5 , we may now bound $|\tilde{\mu}_n(Q) - \nu_n(Q)|$ as follows:

$$(7.14) \quad |\tilde{\mu}_n(Q) - \nu_n(Q)| \leq c(p, d) \frac{\ell^d}{n} + c(d) \frac{1}{\ell} + |\underline{L}''| \max_{\underline{B}(x) \in \underline{L}''} \left(\left| \frac{|\mathbf{C}_\infty \cap \underline{B}^-(x)|}{n^d} - \frac{\theta_p(d) \mathcal{L}^d(\underline{B}^-(x))}{n^d} \right| \right)$$

$$(7.15) \quad \leq c(p, d) \frac{\ell^d}{n} + c(d) \frac{1}{\ell} + \frac{|\underline{L}''|}{n^d} (2\mathcal{L}^d(\underline{B}^-(x))\delta)$$

$$(7.16) \quad \leq c(p, d) \frac{\ell^d}{n} + c(d) \frac{1}{\ell} + c(d)\delta$$

where we have used the bound $|\underline{L}''| \leq |\underline{L}| \leq c(d)(n/\ell)^d$ in going from the second line to the third line. By choosing n sufficiently large, we have $|\tilde{\mu}_n(Q) - \nu_n(Q)| \leq c(p, d)\delta$, and the proof is complete after using the bounds for events $\mathcal{E}_1, \dots, \mathcal{E}_5$. \square

We combine the preceding result with Lemma 6.15 to establish \mathfrak{d} -closeness of μ_n and ν_n with high probability. The following is the central theorem of this section.

Theorem 7.6. *Let $d \geq 3$, $p > p_c(d)$ and let $\delta > 0$. There exist positive constants $c_1(p, d, \delta), c_1(p, d, \delta)$ so that*

$$\mathbb{P}_p \left(\max_{G_n \in \mathcal{G}_n} \mathfrak{d}(\mu_n, \nu_n) \geq \delta \right) \leq c_1 \exp(-c_2 n^{(1-\epsilon(d))/2d})$$

Proof. Let $\delta > 0$ and let Δ^k denote the collection of dyadic cubes in $[-1, 1]^d$ at scale k . There should be no confusion between the k used for dyadic scales, and the renormalization parameter from Section 6, as the latter has been fixed since the end of Section 6. Let $Q \in \Delta^k$. Thanks to Lemma 6.15 and Proposition 7.5, there exist positive constants $c_1(p, d, \delta), c_2(p, d, \delta)$ so that when n is taken sufficiently large depending on δ ,

$$(7.17) \quad \mathbb{P} \left(\max_{G_n \in \mathcal{G}_n} |\mu_n(Q) - \nu_n(Q)| < \delta \right) \geq 1 - c_1 \exp(-c_2 n^{(1-\epsilon(d))/2d})$$

Take j to be large enough so that $2^{-j} < \delta$, and we let Q_1, \dots, Q_m enumerate all dyadic cubes at scales between 0 and $j-1$. Note that the number m of these cubes depends only on δ and d . Let \mathcal{E}_i be the high probability event corresponding to (7.17) for the cube Q_i . We work within the event $\mathcal{E} := \bigcap_{i=1}^m \mathcal{E}_i$, so that by definition of the metric \mathfrak{d} ,

$$(7.18) \quad \mathfrak{d}(\mu_n, \nu_n) \leq \sum_{k=0}^{j-1} \frac{1}{2^k} \sum_{Q \in \Delta^k} \frac{1}{2^{dk}} |\mu_n(Q) - \nu_n(Q)| + \sum_{k=j}^{\infty} \frac{1}{2^k} \sum_{Q \in \Delta^k} \frac{1}{2^{dk}} |\mu_n(Q) - \nu_n(Q)|$$

$$(7.19) \quad \leq 2\delta + \sum_{k=j}^{\infty} \frac{1}{2^k} \sum_{Q \in \Delta^k} \frac{1}{2^{dk}} |\mu_n(Q) - \nu_n(Q)|$$

But note that for each dyadic cube Q , we have the crude bounds $\mu_n(Q) \leq c(d)$ and $\nu_n(Q) \leq c(d)$. Through our choice of j , the second term above is bounded by $c(d)\delta$. Thus, within the event \mathcal{E} , we have $\mathfrak{d}(\mu_n, \nu_n) \leq c(d)\delta$. As m depends only on ϵ and d , the proof is complete. \square

We explore some consequences of Theorem 7.6 before moving to the final section of the paper.

7.2. Closeness to sets of finite perimeter. Recall from Section 2.3 that \mathcal{B}_d is the weak ball about the zero measure of radius 3^d equipped with the metric \mathfrak{d} . For $\gamma, \xi > 0$, define the following collection of measures in \mathcal{B}_d .

$$(7.20) \quad \mathcal{P}_{\gamma, \xi} := \left\{ \nu_F : F \subset [-1, 1]^d, \text{per}(F) \leq \gamma, \mathcal{L}^d(F) \leq \mathcal{L}^d((1 + \xi)W_{p,d}) \right\}$$

We can now easily show that each μ_n is \mathfrak{d} -close to this set.

Corollary 7.7. *Let $d \geq 3$, $p > p_c(d)$ and let $\delta > 0$. There exist positive constants $c_1(p, d, \delta, \xi)$, $c_2(p, d, \delta, \xi)$ and $\gamma(p, d)$ so that*

$$\mathbb{P}_p \left(\max_{G_n \in \mathcal{G}_n} \mathfrak{d}(\mu_n, \mathcal{P}_{\gamma, \xi}) \geq \delta \right) \leq c_1 \exp \left(-c_2 n^{(1-\epsilon(d))/2d} \right)$$

Proof. Let $\delta, \delta', \xi > 0$ and let $\gamma(p, d)$ be as in Corollary 7.4. Let us work within the intersection of the following high probability events

$$(7.21) \quad \mathcal{E}_1 := \left\{ \max_{G_n \in \mathcal{G}_n} \text{per}(P_n) < \gamma n^{d-1} \right\} \quad \mathcal{E}_2 := \left\{ \max_{G_n \in \mathcal{G}_n} \mathfrak{d}(\mu_n, \nu_n) < \min(\delta, \delta') \right\}$$

$$(7.22) \quad \mathcal{E}_3 := \left\{ \frac{|\mathbf{C}_n|}{(2n)^d} \in (\theta_p(d) - \delta', \theta_p(d) + \delta') \right\}$$

respectively from Corollary 7.4, Theorem 7.6 and Corollary A.4. As a consequence of working within \mathcal{E}_2 , we have for each $G_n \in \mathcal{G}_n$,

$$(7.23) \quad \theta_p(d) \mathcal{L}^d(P_n) < \delta' n^d + |G_n|$$

$$(7.24) \quad < \delta' n^d + |\mathbf{C}_n|/d!$$

From working within \mathcal{E}_3 , we may conclude

$$(7.25) \quad \mathcal{L}^d(P_n) < n^d \left(\frac{\delta'}{\theta_p(d)} + \frac{2^d}{d!} \left(1 + \frac{\delta'}{\theta_p(d)} \right) \right)$$

$$(7.26) \quad < n^d \left(\mathcal{L}^d((1 + \xi)W_{p,d}) \right)$$

for δ' chosen sufficiently small according to p, d and ξ . Finally, because we are working within \mathcal{E}_1 , we conclude that $\nu_n \in \mathcal{P}_{\gamma, \xi}$ for each $G_n \in \mathcal{G}_n$. \square

The final observation of this section is that $\mathcal{P}_{\gamma, \xi}$ is \mathfrak{d} -compact.

Lemma 7.8. *The collection of measures $\mathcal{P}_{\gamma, \xi}$ is compact subset of the metric space $(\mathcal{B}_d, \mathfrak{d})$.*

Proof. By Banach-Alaoglu, $(\mathcal{B}_d, \mathfrak{d})$ is a compact metric space. It thus suffices to show $\mathcal{P}_{\gamma, \xi}$ is sequentially closed. Let $\{\nu_{F_k}\}_{k=1}^\infty$ be a sequence of measures in $\mathcal{P}_{\gamma, \xi}$ which converge with respect to the metric \mathfrak{d} . By definition of \mathfrak{d} , it follows that the associated sequence of Borel sets $\{F_k\}_{k=1}^\infty$ converge with respect to the L^1 -metric. Let $F \subset [-1, 1]^d$ be the L^1 -limit of the F_k , and use the definition of \mathfrak{d} once more to conclude ν_F is the \mathfrak{d} -limit of the sequence ν_{F_k} . By Fatou's lemma and the lower semicontinuity of the perimeter functional (Lemma A.13), it follows that both

$$(7.27) \quad \mathcal{L}^d(F) \leq \liminf_{k \rightarrow \infty} \mathcal{L}^d(F_k) \quad \text{and} \quad \text{per}(F) \leq \liminf_{k \rightarrow \infty} \text{per}(F_k)$$

Thus $\nu_F \in \mathcal{P}_{\gamma, \xi}$ and the proof is complete. \square

We are now equipped to prove the main theorems of the paper.

8. PROOF OF MAIN RESULTS

We will prove the main theorems of the paper in this section. Our strategy is outlined as follows: we first use Corollary 7.7 to anchor subsequences of our empirical measures μ_n near sets of finite perimeter. Using a covering lemma for sets of finite perimeter, Lemma 8.1, we will then capture most of the boundary of this set of finite perimeter within a finite collection of balls. Within these balls, we will be able to operate locally on $\partial^\omega G_n$ and apply concentration estimates from Theorem 4.6. The core ideas of this final argument have much in common with the argument given by Cerf and Th  ret in Section 6 in the paper [22]. Throughout this section, γ is fixed and is the constant from Corollary 7.7.

8.1. Setup, the reduced boundary and a covering lemma. We now introduce the reduced boundary of a set of finite perimeter and exhibit a useful covering lemma. Throughout this section, we let α_d denote the volume of the d -dimensional Euclidean unit ball. Given a closed Euclidean ball $B(x, r)$ and a unit vector $v \in \mathbb{S}^{d-1}$, we let $B_-(x, r, v) = \{y \in \mathbb{R}^d : (y - x) \cdot v \leq 0\}$ denote the *lower half ball* of $B(x, r)$ in the direction v .

For $E \subset \mathbb{R}^d$ Borel, we denote by $\nabla \mathbf{1}_E$ the distributional derivative of the indicator function $\mathbf{1}_E$. This is a vector-valued measure whose total variation $\|\nabla \mathbf{1}_E\|$ is the perimeter of E . For $E \subset \mathbb{R}^d$ a set of finite perimeter, the *reduced boundary* $\partial^* E$ of E is the set of points $x \in \mathbb{R}^d$ such that

(i) $\|\nabla \mathbf{1}_E\|(B(x, r)) > 0$ for any $r > 0$.

(ii) If we define $v_r(x) := -\nabla \mathbf{1}_E(B(x, r)) / \|\nabla \mathbf{1}_E\|(B(x, r))$, then as $r \rightarrow 0$, the sequence $v_r(x)$ tends to a limiting unit vector $v_E(x)$, which we call the *exterior normal* to E at x .

The following covering lemma, which we state for general surface energies, is essential to the arguments in this section. Its proof may be found in Section 14.3 of [18].

Lemma 8.1. *Let $F \subset \mathbb{R}^d$ be a set of finite perimeter, and let τ be a norm on \mathbb{R}^d . For any $\delta, \tilde{\delta} > 0$, there exists a finite collection of disjoint balls $\{B(x_i, r_i)\}_{i=1}^m$ such that for each i , we have $x_i \in \partial^* F$, $r_i \in (0, 1)$ and*

$$\mathcal{L}^d[F \cap B(x_i, r_i) \Delta B_-(x_i, r_i, v_F(x_i))] \leq \delta \alpha_d r_i^d$$

$$\left| \mathcal{I}_\tau(F) - \sum_{i=1}^m \alpha_{d-1} r_i^{d-1} \tau(v_F(x_i)) \right| \leq \tilde{\delta}$$

For a set $F \subset [-1, 1]^d$ of perimeter at most γ , and for a ball $B(x, r)$ with $x \in \partial^* F$, we will abbreviate the lower half ball $B_-(x, r, v_F(x))$ as $B_-(x, r)$. Let $\{B(x_i, r_i)\}_{i=1}^m$ be a collection of balls as in Lemma 8.1 for F , for the norm $\beta_{p,d}$ and for the parameters $\delta, \tilde{\delta} > 0$.

Let $\epsilon_F := \delta \min_{i=1}^m \alpha_d(r_i)^d$, so that ϵ_F depends on both δ and $\tilde{\delta}$. Let λ be positive with $\lambda < \mathcal{I}_{p,d}(F)$, so that there is $s \in (0, 1/2)$ with $\lambda \leq \mathcal{I}_{p,d}(F)(1 - 2s)$. We now choose $\tilde{\delta} = \frac{s}{4} \mathcal{I}_{p,d}(F)$. From our choice of the collection $\{B(x_i, r_i)\}_{i=1}^m$, this implies

$$(8.1) \quad \left| \mathcal{I}_{p,d}(F) - \sum_{i=1}^m \alpha_{d-1}(r_i)^{d-1} \beta_{p,d}(v_i) \right| \leq \frac{s}{2} \left(\sum_{i=1}^m \alpha_{d-1}(r_i)^{d-1} \beta_{p,d}(v_i) \right)$$

and hence that

$$(8.2) \quad \lambda \leq (1 - s) \left(\sum_{i=1}^m \alpha_{d-1}(r_i)^{d-1} \beta_{p,d}(v_i) \right)$$

From these computations, we obtain the following lemma.

Lemma 8.2. Let $d \geq 3$ and let $p > p_c(d)$. For $F \subset [-1, 1]^d$ a set of perimeter at most γ , and for the collection $\{B(x_i, r_i)\}_{i=1}^m$ constructed above, we have for each $G_n \in \mathcal{G}_n$,

$$\begin{aligned} \mathbb{P}_p(|\partial^\omega G_n| \leq \lambda n^{d-1} \text{ and } \mathfrak{d}(\mu_n, \nu_F) \leq \epsilon_F) \\ \leq \sum_{i=1}^m \mathbb{P}_p(|\partial^\omega G_n \cap nB(x_i, r_i)| \leq (1-s)n^{d-1}\alpha_{d-1}(r_i)^{d-1}\beta_{p,d}(v_i) \text{ and } \mathfrak{d}(\mu_n, \nu_F) \leq \delta\alpha_d(r_i)^d) \end{aligned}$$

Proof. We first use the disjointness of the balls in the collection $\{B(x_i, r_i)\}_{i=1}^m$, the bound (8.2) on λ and then a union bound.

$$(8.3) \quad \mathbb{P}_p(|\partial^\omega G_n| \leq \lambda n^{d-1} \text{ and } \mathfrak{d}(\mu_n, \nu_F) \leq \epsilon_F)$$

$$(8.4) \quad \leq \mathbb{P}_p\left(\sum_{i=1}^m |\partial^\omega G_n \cap nB(x_i, r_i)| \leq (1-s)n^{d-1} \sum_{i=1}^m \alpha_{d-1}(r_i)^{d-1}\beta_{p,d}(v_i) \text{ and } \mathfrak{d}(\mu_n, \nu_F) \leq \epsilon_F\right)$$

$$(8.5) \quad \leq \sum_{i=1}^m \mathbb{P}_p(|\partial^\omega G_n \cap nB(x_i, r_i)| \leq (1-s)n^{d-1}\alpha_{d-1}(r_i)^{d-1}\beta_{p,d}(v_i) \text{ and } \mathfrak{d}(\mu_n, \nu_F) \leq \epsilon_F)$$

We complete the proof using the definition of ϵ_F . □

Our last task of this subsection is to introduce another small parameter $h > 0$, to be chosen later. For each of the radii r_i , let $r'_i = r_i(1 - h^2)^{1/2}$, so that $\text{cyl}(D(x_i, r'_i), hr'_i) \subset B(x_i, r_i)$. Here, each disc $D(x_i, r'_i)$ is implicitly oriented so that $\text{hyp}(D(x_i, r'_i))$ is orthogonal to v_i .

8.2. Local modification of each $\partial^\omega G_n$. We continue to think of $F \subset [-1, 1]^d$ as fixed, and we also work with a fixed $G_n \in \mathcal{G}_n$. For the collection of balls $\{B(x_i, r_i)\}_{i=1}^m$ from the previous subsection, we show that within each ball $B(x_i, r_i)$, we can find a collection of edges whose size we can control, and so that this collection along with $\partial^\omega G_n$ separates the top and bottom faces of the cylinder $\text{cyl}(D(x_i, r'_i), hr'_i)$.

Let us first fix a ball $B(x_i, r_i)$ within $\{B(x_i, r_i)\}_{i=1}^m$, denoting this ball as $B(x, r)$. Let $J_n = J_n(\omega)$ be the collection of open edges having non-trivial intersection with $N_{5d}(nD(x, r))$. If we close each edge in J_n and each edge in $\partial^\omega G_n$, we break $\mathbb{C}_\infty \cap nB(x, r)$ into a finite number of connected components. We say that one of these components Λ is *outward* if it is contained in $nB(x, r) \setminus nB_-(x, r)$ and *inward* if it is contained in $nB_-(x, r)$.

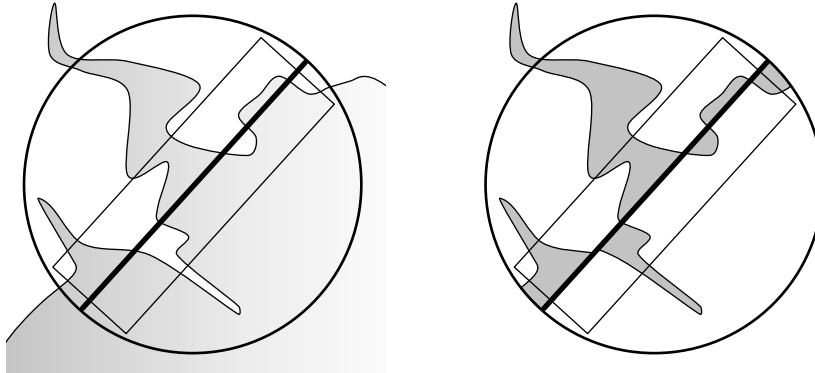


FIGURE 9. The thin cylinder $ncyl(D(x, r'), hr')$ is drawn as a rectangle, the central disc $D(x, r)$ is the bold line. On the left, a generic optimizer G_n viewed up close. On the right, inward and outward components are in grey (outward components point up and to the left). There are three good components and three bad components.

We are only interested in Λ containing vertices incident to edges in J_n . We enumerate all such components which are outward as $\Lambda_1^+, \dots, \Lambda_{\ell^+}^+$, and all such components which are inward as $\Lambda_1^-, \dots, \Lambda_{\ell^-}^-$. Let us say that a component (outward or inward) is *good* if it is contained in $\text{ncyl}(D(x, r), hr')$ and say that it is *bad* otherwise. Here, r' is defined to be $r(1 - h^2)^{1/2}$. Every outward component is a subgraph of G_n ; thus the word “outward” should be understood as relative to the bottom half ball $nB_-(x, r)$. See Figure 9 for an illustration of inward and outward components.

The following lemma tells us that each bad component may be truncated in an efficient way.

Lemma 8.3. *For each bad outward component Λ_j^+ , there is $h_j^+ \in [0, h/2]$ so that*

$$|E^\omega(\Lambda_j^+) \cap (nD(x, r) + nh_j^+ r' v_F(x))| \leq c(d) \frac{|\Lambda_j^+|}{nhr}$$

Likewise, for each bad inward component Λ_j^- , there is $h_j^- \in [0, h/2]$ so that

$$|E^\omega(\Lambda_j^-) \cap (nD(x, r) - nh_j^- r' v_F(x))| \leq c(d) \frac{|\Lambda_j^-|}{nhr}$$

Proof. Let us first handle the claim for outward components Λ_j^+ . For $\alpha \in [0, h/2]$, define the edge set

$$(8.6) \quad B_{j,\alpha}^+ := E^\omega(\Lambda_j^+) \cap (nD(x, r) + n\alpha r' v_F(x))$$

For $k \in \{1, \dots, \lceil nh/2 \rceil\}$, let $\alpha_k = k/2n$. We have

$$(8.7) \quad \bigcup_{k=1}^{\lceil nh/2 \rceil} B_{j,\alpha_k}^+ \subset E^\omega(\Lambda_j^+)$$

Whenever k and k' are such that $|k - k'| \geq 10d$, we also have that B_{j,α_k}^+ and $B_{j,\alpha_{k'}}^+$ are disjoint. Thus,

$$(8.8) \quad \sum_{k=1}^{\lceil nh/2 \rceil} |B_{j,\alpha_k}^+| \leq (10d) |E^\omega(\Lambda_j^+)| \leq c(d) |\Lambda_j^+|$$

As $r < 1$, for one such k we must have

$$(8.9) \quad |B_{j,\alpha_k}^+| \leq \frac{c(d) |\Lambda_j^+|}{nhr}$$

By construction, this α_k is at most $h/2$. For this k , which depends on the index of the outward component j , we relabel B_{j,α_k}^+ as B_j^+ and set $h_j^+ = \alpha_k$. Analogous reasoning for bad inward components gives the existence of $h_j^- \in [0, h/2]$ for each bad inward Λ_j^- so that

$$(8.10) \quad |B_{j,h_j^-}^-| \leq c(d) \frac{|\Lambda_j^-|}{nhr}$$

where

$$(8.11) \quad B_{j,\alpha}^- := E(\Lambda_j^-) \cap (nD(x, r) - n\alpha r' v_F(x))$$

and we relabel B_{j,h_j^-} as B_j^- for brevity. This completes the proof. \square

We will continue to refer to B_j^+ and B_j^- in the rest of this subsection, so let us define these edge sets once more now that we have shown the existence of h_j^\pm . For bad Λ_j^\pm , we define

$$(8.12) \quad B_j^\pm := E^\omega(\Lambda_j^\pm) \cap (nD(x, r) \pm nh_j^\pm r' v_F(x))$$

If Λ_j^\pm is good, we set B_j^\pm to be empty. As each h_j^+ and h_j^- lies within $[0, h/2]$, it follows that each collection of edges B_j^+ and B_j^- is contained within $\text{ncyl}(D(x, r), hr') \cap nB(x, r)$. Figure 10 depicts the construction of the B_j^\pm in the context of setup from Figure 9. The following is an immediate consequence of Lemma 8.3.

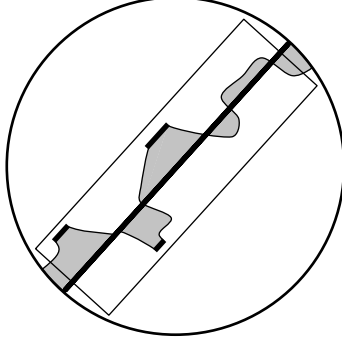


FIGURE 10. The bold lines parallel to the central disc are the efficiently chosen edge sets B_j^\pm for bad components. Together, with part of $\partial^\omega G_n$, they form an open cutset separating the top and bottom faces of the thin cylinder $\text{ncyl}(D(x, r'), hr')$ in \mathbf{C}_∞ .

Corollary 8.4. *For the edge sets B_j^+ and B_j^- defined above, we have*

$$\sum_{j=1}^{\ell^+} |B_j^+| + \sum_{j=1}^{\ell^-} |B_j^-| \leq \frac{c(d)}{nhr} \left(\sum_{j=1}^{\ell^+} |\Lambda_j^+| + \sum_{j=1}^{\ell^-} |\Lambda_j^-| \right)$$

Whenever some $G_n \in \mathcal{G}_n$ is such that the associated μ_n is \mathfrak{d} -close to ν_F , and whenever $B(x, r)$ is a ball corresponding to a covering of $\partial^* F$ as in Lemma 8.1, we may bound the total size of the inward and outward components within this ball.

Proposition 8.5. *Let $G_n \in \mathcal{G}_n$, and consider the associated empirical measure μ_n . Let $\delta > 0$, and suppose that for F a set of finite perimeter and $B(x, r)$ a ball with $x \in \partial^* F$ we have both*

$$\mathfrak{d}(\mu_n(\omega), \mu_F) \leq \delta \alpha_d r^d \quad \text{and} \quad \mathcal{L}^d((B(x, r) \cap F) \Delta B_-(x, r)) \leq \delta \alpha_d r^d$$

Then for the Λ_j^\pm defined above, and for n taken sufficiently large depending on r, δ, p and d , we have

$$\sum_{j=1}^{\ell^+} |\Lambda_j^+| + \sum_{j=1}^{\ell^-} |\Lambda_j^-| \leq c(p, d) n^d \delta \alpha_d r^d$$

Proof. Let us write $\theta_p(d)$ as θ in this proof. We first work with the outward components. If Λ_j^+ is an outward component, it is contained in $G_n \cap (B(x, r) \setminus B_-(x, r))$. Thus, from the definition of \mathfrak{d} , we have that

$$(8.13) \quad \sum_{i=1}^{\ell^+} |\Lambda_i^+| \leq n^d \mathfrak{d}(\mathbf{1}_{B(x, r)} \mu_n(\omega), \nu_{B_-(x, r)})$$

where $\mathbf{1}_{B(x, r)} \mu_n(\omega)$ denotes the restriction of $\mu_n(\omega)$ to the ball $B(x, r)$. From the hypothesis that $\mathcal{L}^d((F \cap B(x, r)) \Delta B_-(x, r)) \leq \delta \alpha_d r^d$ we may again use the definition of \mathfrak{d} to deduce that,

$$\mathfrak{d}(\nu_{B_-(x, r)}, \nu_{F \cap B(x, r)}) \leq \theta \delta \alpha_d r^d$$

By the triangle inequality,

$$(8.14) \quad \sum_{i=1}^{\ell^+} |\Lambda_j^+| \leq n^d \left(\mathfrak{d}(\mathbf{1}_{B(x,r)} \mu_n(\omega), \nu_{F \cap B(x,r)}) + \theta \delta \alpha_d r^d \right)$$

$$(8.15) \quad \leq n^d \left(\mathfrak{d}(\mu_n(\omega), \nu_F) + \theta \delta \alpha_d r^d \right)$$

$$(8.16) \quad \leq 2n^d \delta \alpha_d r^d$$

where we've used the hypothesis that $\mathfrak{d}(\mu_n(\omega), \nu_F) < \delta \alpha_d r^d$. This handles the claim for the outward components. For Λ_j^- an inward component, we have $\Lambda_j^- \subset B_-(x, r) \setminus G_n$. Recall that for each $y \in \mathbb{Z}^d$, we defined the dual cube $Q(y)$ as $y + [-1/2, 1/2]^d$. For each Λ_j^- , define

$$(8.17) \quad \text{poly}(\Lambda_j^-) := \bigcup_{y \in \Lambda_j^-} Q(y)$$

and observe that each $\text{poly}(\Lambda_j^-)$ is almost contained within $nB_-(x, r) \setminus G_n$. More specifically, the only dual cubes of $\text{poly}(\Lambda_j^-)$ which are not contained in $nB_-(x, r) \setminus G_n$ are those which intersect the equatorial disc $nD(x, r)$. Thus, each $\text{poly}(\Lambda_j^-)$ is contained within the union $(B_-(x, r) \setminus G_n) \cup \mathcal{N}_{c(d)}(nD(x, r))$, with $c(d)$ chosen appropriately. It follows that

$$(8.18) \quad \bigcup_{j=1}^{\ell^-} \text{poly}(\Lambda_j^-) \subset (B_-(x, r) \setminus G_n) \cup \mathcal{N}_{c(d)}(nD(x, r))$$

As distinct $\text{poly}(\Lambda_j^-)$ are almost disjoint, we may conclude

$$(8.19) \quad \sum_{j=1}^{\ell^-} |\Lambda_j^-| = \mathcal{L}^d \left(\bigcup_{j=1}^{\ell^-} \text{poly}(\Lambda_j^-) \right)$$

$$(8.20) \quad \leq \theta^{-1} \mathfrak{d}(\mathbf{1}_{B(x,r)} \mu_n(\omega), \nu_{B_-(x,r)}) + \mathcal{L}^d(\mathcal{N}_{c(d)}(nD(x, r)))$$

Where the second line is obtained from the first line by using the definition of \mathfrak{d} and (8.18). Using our bounds on $\mathfrak{d}(\mathbf{1}_{B(x,r)} \mu_n(\omega), \nu_{B_-(x,r)})$ from (8.16), we deduce

$$(8.21) \quad \sum_{j=1}^{\ell^-} |\Lambda_j^-| \leq c(p, d) n^d \delta \alpha_d r^d + c(d) n^{d-1}$$

And the proof is complete upon taking n sufficiently large depending on r, δ, p and d . \square

We now return to the overing of $\partial^* F$ initially constructed. For each ball $B(x_i, r_i)$, let $E_n^{(i)}$ be the union of $\partial^\omega G_n \cap nB(x_i, r_i)$ with all edge sets B_j^\pm corresponding to the ball $B(x_i, r_i)$. Note that each $E_n^{(i)}$ depends on n , the ball $B(x_i, r_i)$, $G_n \in \mathcal{G}_n$ and of course the percolation configuration ω . Putting Proposition 8.5 together with Corollary 8.4, we obtain the following.

Corollary 8.6. *Let $F \subset [-1, 1]^d$ be a set of perimeter at most γ , and let $G_n \in \mathcal{G}_n$. For each ball $B(x_i, r_i) \in \{B(x_i, r_i)\}_{i=1}^m$, and for n sufficiently large in a way depending on r_i, δ, p and d , we have*

$$\begin{aligned} |\partial^\omega G_n \cap nB(x_i, r_i)| &\leq (1-s) n^{d-1} \alpha_{d-1}(r_i)^{d-1} \beta_{p,d}(v_i) \text{ and } \mathfrak{d}(\mu_n, \nu_F) \leq \delta \alpha_d (r_i)^d \\ &\subset \left\{ \left| E_n^{(i)} \right| \leq \left(1-s + c(p, d) \frac{\delta}{h} \right) n^{d-1} \alpha_{d-1}(r_i)^{d-1} \beta_{p,d}(v_i) \right\} \end{aligned}$$

Proof. Let $G_n \in \mathcal{G}_n$ and work within the event

$$(8.22) \quad \{|\partial^\omega G_n \cap nB(x_i, r_i)| \leq (1-s)n^{d-1}\alpha_{d-1}(r_i)^{d-1}\beta_{p,d}(v_i) \text{ and } \mathfrak{d}(\mu_n, \nu_F) \leq \delta\alpha_d(r_i)^d\}$$

and by the hypotheses on the collection $\{B(x_i, r_i)\}_{i=1}^m$, for each ball in this collection we have

$$(8.23) \quad \mathfrak{d}(\mu_n(\omega), \mu_F) \leq \delta\alpha_d r^d \quad \text{and} \quad \mathcal{L}^d((B(x_i, r_i) \cap F)\Delta B_-(x_i, r_i)) \leq \delta\alpha_d(r_i)^d$$

Thus we are in a position to apply Proposition 8.5 to each ball, so that for n sufficiently large depending on r_i, δ, p and d ,

$$(8.24) \quad \sum_{j=1}^{\ell^+} |\Lambda_j^+| + \sum_{j=1}^{\ell^-} |\Lambda_j^-| \leq c(p, d)n^d \delta\alpha_d(r_i)^d$$

Where the Λ_j^\pm are understood to be the inward and outward components corresponding to the ball $B(x_i, r_i)$. Applying Corollary 8.4, we find that for the edge sets B_j^\pm corresponding to the ball $B(x_i, r_i)$, we have

$$(8.25) \quad \sum_{j=1}^{\ell^+} |B_j^+| + \sum_{j=1}^{\ell^-} |B_j^-| \leq \frac{c(p, d)}{h} n^{d-1} \delta\alpha_{d-1}(r_i)^{d-1}$$

The proof is complete, as $E_n^{(i)}$ is precisely the union of the edge sets B_j^\pm with $\partial^\omega G_n \cap nB(x_i, r_i)$. \square

The next lemma tells us that with high probability, each $E_n^{(i)}$ forms an open cutset. Let $r_F = \min_{i=1}^m r_i$, so that $r_F = r_F(\lambda, \delta, p, d)$, by virtue of the fact that $\tilde{\delta}$ depends on $s = s(F, \lambda, p, d)$. Let $r'_F = r_F(1 - h^2)^{1/2}$.

Lemma 8.7. *Let \mathcal{E} be the event that for each $G_n \in \mathcal{G}_n$, and for each i , any path of open edges in the discrete cylinder $\text{d-cyl}(D(x, r'_i), hr'_i, n)$ between the faces $\text{d-face}^\pm(D(x, r'_i), hr'_i, n)$ must use an edge of $E_n^{(i)}$. There exist positive constants $c_1(p, d)$ and $c_2(p, d)$ so that*

$$\mathbb{P}_p(\mathcal{E}) \geq 1 - c_1 \exp(-c_2(r'_F hn)^{(d-1)/d})$$

In particular, for each configuration $\omega \in \mathcal{E}$, by completing each $E_n^{(i)}$ to a full cutset using only closed edges, we may conclude that $|E_n^{(i)}| \geq \mathfrak{X}_{\text{face}}(D(x_i, r'_i), hr'_i, n)$ in the configuration ω .

Proof. Our primary tool is Theorem A.5. Let us drop the indexing for the sake of clarity and work with generic objects: balls $B(x, r)$, discs $D(x, r')$ and edge sets E_n .

From the definition of the B_j^\pm , any open path in C_∞ between the faces $\text{d-face}^\pm(D(x, r'), hr', n)$ within $\text{d-cyl}(D(x, r'), hr', n)$ must use an edge of E_n . As we are working within the almost sure event that there is a unique infinite cluster, the only way the faces $\text{d-face}^\pm(D(x, r'), hr', n)$ may be joined by an open path in $\text{d-cyl}(D(x, r'), hr', n)$ is if this open path lies in a finite cluster. Such a path must use at least $2r'hn$ edges, so that the cluster containing this path must have volume at least $2r'hn$. We use a union bound with Theorem A.5 applied to each point in $[-n, n]^d \cap \mathbb{Z}^d$ to obtain the desired result. \square

The following proposition aggregates all the work of the previous two subsections. Recall that ϵ_F was defined as $\delta \min_{i=1}^m \alpha_d(r_i)^d$.

Proposition 8.8. *Let $d \geq 3$ and let $p > p_c(d)$. Let $F \subset [-1, 1]^d$ be a Borel set of perimeter at most γ , let $G_n \in \mathcal{G}_n$ and let $\lambda > 0$ be given with $\lambda < \mathcal{I}_{p,d}(F)$. There exist positive constants $c_1(\lambda, F, p, d)$, $c_2(\lambda, F, p, d)$ and $\epsilon_F(\lambda, F, p, d)$ so that*

$$\mathbb{P}_p(\exists G_n \in \mathcal{G}_n \text{ such that } |\partial^\omega G_n| \leq \lambda n^{d-1} \text{ and } \mathfrak{d}(\mu_n, \nu_F) \leq \epsilon_F) \leq c_1 \exp(-c_2 n^{1/2d})$$

Proof. Let $G_n \in \mathcal{G}_n$. Recall that s was chosen so that $\lambda \leq \mathcal{I}_{p,d}(F)(1-2s)$. We begin by using Lemma 8.2, Corollary 8.6 and Lemma 8.7 together to obtain that when n is sufficiently large depending on r_F, δ, p, d ,

$$(8.26) \quad \mathbb{P}_p(\exists G_n \in \mathcal{G}_n \text{ such that } |\partial^\omega G_n| \leq \lambda n^{d-1} \text{ and } \mathfrak{d}(\mu_n, \nu_F) \leq \epsilon_F)$$

$$(8.27) \quad \leq \sum_{i=1}^m \mathbb{P}_p \left(\mathfrak{X}_{\text{face}}(D(x_i, r'_i), hr'_i, n) \leq \left(1 - s + c(p, d) \frac{\delta}{h}\right) n^{d-1} \alpha_{d-1}(r_i)^{d-1} \beta_{p,d}(\nu_i) \right)$$

$$(8.28) \quad + c_1 \exp(-c_2(r'_F hn)^{(d-1)/d})$$

Now, choose $\delta = \delta(s, h, p, d)$ so that

$$(8.29) \quad \frac{1}{(1-h^2)^{(d-1)/2}} \left(1 - s + c(p, d) \frac{\delta}{h}\right) \leq 1 - \frac{s}{2}$$

For δ chosen this way,

$$(8.30) \quad \mathbb{P}_p(\exists G_n \in \mathcal{G}_n \text{ such that } |\partial^\omega G_n| \leq \lambda n^{d-1} \text{ and } \mathfrak{d}(\mu_n, \nu_F) \leq \epsilon_F)$$

$$(8.31) \quad \leq \sum_{i=1}^m \mathbb{P}_p \left(\mathfrak{X}_{\text{face}}(D(x_i, r'_i), hr'_i, n) \leq \left(1 - \frac{s}{2}\right) n^{d-1} \alpha_{d-1}(r'_i)^{d-1} \beta_{p,d}(\nu_i) \right)$$

$$(8.32) \quad + c_1 \exp(-c_2(r'_F hn)^{(d-1)/d})$$

This may be re-written as follows:

$$(8.33) \quad \mathbb{P}_p(\exists G_n \in \mathcal{G}_n \text{ such that } |\partial^\omega G_n| \leq \lambda n^{d-1} \text{ and } \mathfrak{d}(\mu_n, \nu_F) \leq \epsilon_F)$$

$$(8.34) \quad \leq \sum_{i=1}^m \mathbb{P}_p \left(\mathfrak{X}_{\text{face}}(D(x_i/r'_i, 1), h, r'_i n) \leq \left(1 - \frac{s}{2}\right) \mathcal{H}^{d-1}(r'_i n D(x_i, 1)) \beta_{p,d}(\nu_i) \right)$$

$$(8.35) \quad + c_1 \exp(-c_2(r'_F hn)^{(d-1)/d})$$

Choose $h = h(s, p, d)$ sufficiently small so that we may apply Proposition 5.2. We note the slight discrepancy in notation for discs here versus discs in Proposition 5.2. This discrepancy stems from the fact that in Proposition 5.2, the direction of the disc needs to be specified but the radius is assumed to be one, where as in our case the center x_i determines the direction of the disc, but the radius needs to be specified. Applying Proposition 5.2, we see there are positive constants $c_1(s, p, d)$ and $c_2(s, p, d)$ so that

$$(8.36) \quad \mathbb{P}_p(\exists G_n \in \mathcal{G}_n \text{ such that } |\partial^\omega G_n| \leq \lambda n^{d-1} \text{ and } \mathfrak{d}(\mu_n, \nu_F) \leq \epsilon_F)$$

$$(8.37) \quad \leq mc_1 \exp(-c_2(r_F n)^{(d-1)/3}) + c_1 \exp(-c_2(r'_F hn)^{(d-1)/d})$$

$$(8.38) \quad \leq mc_1 \exp(-c_2(r_F n)^{1/2d})$$

where we have replaced r'_F by r_F as we lose no generality assuming h is sufficiently small so that $(1-h^2)^{1/2} \geq 1/2$. The proof is almost complete, we need only keep track of a few dependencies. As $h = h(s, p, d)$, the dependence of δ may be reformulated as $\delta = \delta(s, p, d)$. We also note that $s = s(\lambda, F, p, d)$. This implies the constants δ and $\tilde{\delta}$ also depend only on λ, F, p, d . As $r_F = r_F(F, \delta, \tilde{\delta})$ and because $m = m(\gamma, \delta, \tilde{\delta})$, with γ having been fixed after Corollary 7.7, we conclude there are positive constants $c_1(\lambda, F, p, d)$ and $c_2(\lambda, F, p, d)$ so that

$$(8.39) \quad \mathbb{P}_p(\exists G_n \in \mathcal{G}_n \text{ such that } |\partial^\omega G_n| \leq \lambda n^{d-1} \text{ and } \mathfrak{d}(\mu_n, \nu_F) \leq \epsilon_F) \leq c_1 \exp(-c_2 n^{1/2d})$$

and we are finished. \square

We will use Proposition 8.8 in conjunction with a compactness argument to prove the main results of the paper.

8.3. Proof of main results. We first give the proof of Theorem 2.2, from which we will easily be able to deduce Theorem 1.3 and Theorem 1.2. We then use Theorem 1.2 to deduce Theorem 1.3. Before doing so, we must introduce a quantitative version of the isoperimetric inequality for the anisotropic surface energy $\mathcal{I}_{p,d}$. Given $F \subset \mathbb{R}^d$ a set of finite perimeter, we define the *asymmetry index* (or *Fraenkel asymmetry* in the Euclidean setting) of F as follows:

$$(8.40) \quad A(F) := \inf \left\{ \frac{\mathcal{L}^d(F \Delta (x + rW_{p,d}))}{\mathcal{L}^d(F)} : x \in \mathbb{R}^d, \mathcal{L}^d(rW_{p,d}) = \mathcal{L}^d(F) \right\}$$

For $r > 0$ chosen so that $rW_{p,d}$ and F have the same volume, we also define the *isoperimetric deficit* of F as

$$(8.41) \quad D(F) := \frac{\mathcal{I}_{p,d}(F) - \mathcal{I}_{p,d}(rW_{p,d})}{\mathcal{I}_{p,d}(rW_{p,d})}$$

The anisotropic isoperimetric inequality tells us that $D(F) \geq 0$ for all sets F of finite perimeter. As a consequence of the Taylor's theorem, Theorem 2.3, we have that $D(F) = 0$ if and only if $A(F) = 0$. There is a wonderful quantitative version of this result, due to Figalli, Maggi and Pratelli [31], which we specialize to our surface energy $\mathcal{I}_{p,d}$.

Theorem 8.9. *Let $F \subset \mathbb{R}^d$ be a set of finite perimeter with $\mathcal{L}^d(F) < \infty$. There is a positive constant $c(d)$ so that*

$$A(F) \leq c(d)D(F)^{1/2}$$

As an immediate consequence of this theorem, we have that whenever $rW_{p,d}$ is a dilate of the Wulff crystal, and whenever F^r is a set of finite perimeter such that $\mathcal{L}^d(F^r) = \mathcal{L}^d(rW_{p,d})$, we have

$$(8.42) \quad \frac{\mathcal{I}_{p,d}(F^r)}{\mathcal{I}_{p,d}(rW_{p,d})} \geq 1 + c(d)(A(F^r))^2$$

Let $\xi > 0$ and define $\eta = \eta(\xi)$ via the relation

$$(8.43) \quad (1 - \eta) = \frac{1}{1 + \xi}$$

Define the following subset of collection of measures:

$$(8.44) \quad \mathcal{W}_{p,d,\xi} := \left\{ \nu_{W+x} : \begin{array}{l} x \in \mathbb{R}^d, (W+x) \subset [-1, 1]^d \text{ and } W \text{ is a dilate of } W_{p,d} \\ \text{such that } \mathcal{L}^d((1-\eta)W_{p,d}) \leq \mathcal{L}^d(W) \leq \mathcal{L}^d((1+\xi)W_{p,d}) \end{array} \right\}$$

Proof of Theorem 2.2. Let $\zeta > 0$, and choose $\xi = \xi(\zeta) > 0$ and $\epsilon = \epsilon(\zeta, \xi) > 0$ so that both

$$(8.45) \quad \frac{1}{1 + \xi} = 1 - 2\zeta \quad \text{and} \quad \frac{1 + \xi}{1 + c(p, d)\epsilon^2} = 1 - 2\zeta$$

where $c(p, d)$ shall be specified later. We remark that ϵ is not the same as the parameter $\epsilon(d)$ defined in (6.10), this latter fixed value will only show up in the exponent of an upper bound on a probability. Recall that

$$(8.46) \quad \mathcal{P}_{\gamma,\xi} = \left\{ \nu_F : F \subset [-1, 1]^d, \text{per}(F) \leq \gamma, \mathcal{L}^d(F) \leq \mathcal{L}^d((1 + \xi)W_{p,d}) \right\}$$

As γ was chosen large enough so that $\text{per}(W_{p,d}) \leq \gamma$, we see that $\mathcal{W}_{p,d,\xi} \subset \mathcal{P}_{\gamma,\xi}$ (we can of course take γ slightly larger to account for the increase in perimeter coming from dilating the Wulff crystal). Our principal aim is to show that the probabilities

$$(8.47) \quad \mathbb{P}_p(\exists G_n \in \mathcal{G}_n \text{ such that } \mathfrak{d}(\mu_n, \mathcal{W}_{p,d,\xi}) \geq \epsilon)$$

decay rapidly with n .

Let $\delta, \delta' > 0$ and let $\mathcal{V}_\epsilon(\mathcal{W}_{p,d,\xi})$ denote the open ϵ -neighborhood of $\mathcal{W}_{p,d,\xi}$ with respect to the metric \mathfrak{d} . Let $\mathcal{K}_{\gamma,\xi,\epsilon}$ be the complement of this neighborhood in $\mathcal{P}_{\gamma,\xi}$ which is compact by Lemma 7.8. For each

$\nu_F \in \mathcal{P}_{\gamma, \xi}$, let ϵ_F be as in Proposition 8.8 for F and the parameter $\lambda_F = (1 - \zeta)\mathcal{I}_{p,d}(F)$. We take ϵ_F smaller if necessary depending on δ' , so that

$$(8.48) \quad (1 + \epsilon_F/\theta_p(d)) \leq (1 + \delta')$$

we let $\mathcal{B}(\nu_F, \epsilon_F/2)$ denote the \mathfrak{d} -ball about ν_F of radius $\epsilon_F/2$. As $\mathcal{K}_{\gamma, \xi, \epsilon}$ is compact, the \mathfrak{d} -balls $\{\mathcal{B}(\nu_F, \epsilon_F/2)\}_{\nu_F \in \mathcal{K}_{\gamma, \xi, \epsilon}}$ form an open cover of $\mathcal{K}_{\gamma, \xi, \epsilon}$ which admits a finite subcover. Thus, there are finitely many sets of perimeter at most γ , which we enumerate as F_1, \dots, F_m , such that $\nu_{F_j} \in \mathcal{K}_{\gamma, \xi, \epsilon}$ for each j , and such that $\{\mathcal{B}(\nu_{F_j}, \epsilon_{F_j}/2)\}_{j=1}^m$ forms an open cover of $\mathcal{K}_{\gamma, \xi, \epsilon}$. We now choose δ so that

$$(8.49) \quad \delta \leq \min_{j=1}^m \epsilon_{F_j}/2$$

We now begin to estimate (8.47).

$$(8.50) \quad \mathbb{P}_p(\exists G_n \in \mathcal{G}_n \text{ such that } \mathfrak{d}(\mu_n, \mathcal{W}_{p,d,\xi}) \geq \epsilon)$$

$$(8.51) \quad \leq \mathbb{P}_p\left(\max_{G_n \in \mathcal{G}_n} \mathfrak{d}(\mu_n, \mathcal{W}_{p,d,\xi}) \geq \epsilon \text{ and } n\widehat{\Phi}_n \leq (1 + \delta')\varphi_{W_{p,d}}\right) + \mathbb{P}_p\left(n\widehat{\Phi}_n > (1 + \delta')\varphi_{W_{p,d}}\right)$$

$$(8.52) \quad \leq \mathbb{P}_p\left(\max_{G_n \in \mathcal{G}_n} \mathfrak{d}(\mu_n, \mathcal{W}_{p,d,\xi}) \geq \epsilon \text{ and } n\widehat{\Phi}_n \leq (1 + \delta')\varphi_{W_{p,d}}\right) + c_1 \exp(-c_2 n^{(d-1)/3})$$

Where we have used the bounds in the proof of Corollary 5.6, and where we recall that $\varphi_{W_{p,d}}$ is the continuum version of the conductance (defined at the end of Section 3) for the Wulff crystal. By invoking Corollary 7.7 for the parameter δ , we may further deduce

$$(8.53) \quad \mathbb{P}_p(\exists G_n \in \mathcal{G}_n \text{ such that } \mathfrak{d}(\mu_n, \mathcal{W}_{p,d,\xi}) \geq \epsilon)$$

$$(8.54) \quad \leq \mathbb{P}_p\left(\max_{G_n \in \mathcal{G}_n} \mathfrak{d}(\mu_n, \mathcal{W}_{p,d,\xi}) \geq \epsilon \text{ and } \max_{G_n \in \mathcal{G}_n} \mathfrak{d}(\mu_n, \mathcal{P}_{\gamma, \xi}) < \delta \text{ and } n\widehat{\Phi}_n \leq (1 + \delta')\varphi_{W_{p,d}}\right)$$

$$(8.55) \quad + c_1 \exp(-c_2 n^{(1-\epsilon(d))/2d})$$

Now we apply a union bound and the triangle inequality.

$$(8.56) \quad \mathbb{P}_p(\exists G_n \in \mathcal{G}_n \text{ such that } \mathfrak{d}(\mu_n, \mathcal{W}_{p,d,\xi}) \geq \epsilon)$$

$$(8.57) \quad \leq \sum_{i=1}^m \mathbb{P}_p\left(\exists G_n \in \mathcal{G}_n \text{ such that } \mathfrak{d}(\mu_n, \nu_{F_j}) \leq \epsilon_{F_j}/2 + \delta \text{ and } n\widehat{\Phi}_n \leq (1 + \delta')\varphi_{W_{p,d}}\right)$$

$$(8.58) \quad + c_1 \exp(-c_2 n^{(1-\epsilon(d))/2d})$$

$$(8.59) \quad \leq \sum_{i=1}^m \mathbb{P}_p\left(\exists G_n \in \mathcal{G}_n \text{ such that } \mathfrak{d}(\mu_n, \nu_{F_j}) \leq \epsilon_{F_j} \text{ and } n\widehat{\Phi}_n \leq (1 + \delta')\varphi_{W_{p,d}}\right)$$

$$(8.60) \quad + c_1 \exp(-c_2 n^{(1-\epsilon(d))/2d})$$

where we have used (8.49) to obtain this last line. Let us focus on bounding each summand of the form

$$(8.61) \quad \mathbb{P}_p(\mathcal{F}_j) := \mathbb{P}_p\left(\exists G_n \in \mathcal{G}_n \text{ such that } \mathfrak{d}(\mu_n, \nu_{F_j}) \leq \epsilon_{F_j} \text{ and } n\widehat{\Phi}_n \leq (1 + \delta')\varphi_{W_{p,d}}\right)$$

We begin by unravelling the Cheeger constant.

$$(8.62) \quad \mathbb{P}_p(\mathcal{F}_j) = \mathbb{P}_p\left(\exists G_n \in \mathcal{G}_n \text{ such that } \mathfrak{d}(\mu_n, \nu_{F_j}) \leq \epsilon_{F_j} \text{ and } n|\partial^\omega G_n| \leq (1 + \delta')|G_n|\varphi_{W_{p,d}}\right)$$

$$(8.63) \quad \leq \mathbb{P}_p\left(\exists G_n \in \mathcal{G}_n \text{ such that } \mathfrak{d}(\mu_n, \nu_{F_j}) \leq \epsilon_{F_j} \text{ and } n|\partial^\omega G_n| \leq (1 + \delta')n^d(\theta_p(d)\mathcal{L}^d(F_j) + \epsilon_{F_j})\varphi_{W_{p,d}}\right)$$

$$(8.64) \quad \leq \mathbb{P}_p\left(\exists G_n \in \mathcal{G}_n \text{ such that } \mathfrak{d}(\mu_n, \nu_{F_j}) \leq \epsilon_{F_j} \text{ and } |\partial^\omega G_n| \leq (1 + \delta')^2 n^{d-1} \theta_p(d) \mathcal{L}^d(F_j) \varphi_{W_{p,d}}\right)$$

Here we have used the definition of the metric \mathfrak{d} , as well as our assumptions on the ϵ_F from (8.48). We now define $W_{p,d}^\xi := (1 + \xi)W_{p,d}$, define $W_{p,d}^\eta := (1 - \eta)W_{p,d}$ and observe that

(8.65)

$$\begin{aligned} \mathbb{P}_p(\mathcal{F}_j) &\leq \mathbb{P}_p\left(\exists G_n \in \mathcal{G}_n \text{ such that } \mathfrak{d}(\mu_n, \nu_{F_j}) \leq \epsilon_{F_j} \text{ and } |\partial^\omega G_n| \leq (1 + \delta')^2 n^{d-1} \mathcal{I}_{p,d}(F_j)(\varphi_{F_j})^{-1} \varphi_{W_{p,d}}\right) \\ (8.66) \quad &\leq \mathbb{P}_p\left(\exists G_n \in \mathcal{G}_n \text{ such that } \mathfrak{d}(\mu_n, \nu_{F_j}) \leq \epsilon_{F_j} \text{ and } |\partial^\omega G_n| \leq (1 + \delta')^2 (1 + \xi) n^{d-1} \mathcal{I}_{p,d}(F_j)(\varphi_{F_j})^{-1} \varphi_{W_{p,d}^\xi}\right) \end{aligned}$$

In the case that $\mathcal{L}^d(F_j) \leq \mathcal{L}^d(W_{p,d}^\eta)$, which we refer to as Case I, the relation (8.43) implies

$$(8.67) \quad (\varphi_{F_j})^{-1} \varphi_{W_{p,d}^\xi} \leq (\varphi_{W_{p,d}^\eta})^{-1} \varphi_{W_{p,d}^\xi} \leq \frac{1}{(1 + \xi)^2}$$

Thus, in Case I we may conclude

$$(8.68) \quad \mathbb{P}_p(\mathcal{F}_j) \leq \mathbb{P}_p\left(\exists G_n \in \mathcal{G}_n \text{ such that } \mathfrak{d}(\mu_n, \nu_{F_j}) \leq \epsilon_{F_j} \text{ and } |\partial^\omega G_n| \leq \frac{(1 + \delta')^2}{1 + \xi} n^{d-1} \mathcal{I}_{p,d}(F_j)\right)$$

As ξ was chosen as in (8.45), we may choose δ' small enough in a way depending on ξ and ζ so that

$$(8.69) \quad \mathbb{P}_p(\mathcal{F}_j) \leq \mathbb{P}_p\left(\exists G_n \in \mathcal{G}_n \text{ such that } \mathfrak{d}(\mu_n, \nu_{F_j}) \leq \epsilon_{F_j} \text{ and } |\partial^\omega G_n| \lambda_{F_j} n^{d-1}\right)$$

with $\lambda_{F_j} = (1 - \zeta) \mathcal{I}_{p,d}(F_j)$. This puts us in a position to apply Proposition 8.8. We now maneuver into a similar position in Case II, in which $\mathcal{L}^d(F_j) \geq \mathcal{L}^d(W_{p,d}^\eta)$. That each F_j corresponds to $\nu_{F_j} \in \mathcal{K}_{\gamma,\xi,\epsilon}$ implies that each ν_{F_j} is a \mathfrak{d} -distance at least ϵ to the set $\mathcal{W}_{p,d,\xi}$. It follows from the volume of F_j in Case II and the definition of \mathfrak{d} that the asymmetry index $A(F_j)$ is at least $c(p, d)\epsilon$, where we have lost no generality assuming $\xi \leq 2^d/d!$, for instance. Given F_j in Case II, let F_j^ξ denote the dilate of F_j so that $\mathcal{L}^d(F_j^\xi) = \mathcal{L}^d(W_{p,d}^\xi)$. Note that F_j and F_j^ξ have the same asymmetry index.

(8.70)

$$\begin{aligned} \mathbb{P}_p(\mathcal{F}_j) &\leq \mathbb{P}_p\left(\exists G_n \in \mathcal{G}_n \text{ such that } \mathfrak{d}(\mu_n, \nu_{F_j}) \leq \epsilon_{F_j} \text{ and } |\partial^\omega G_n| \leq (1 + \delta')^2 (1 + \xi) n^{d-1} \mathcal{I}_{p,d}(F_j)(\varphi_{F_j})^{-1} \varphi_{W_{p,d}^\xi}\right) \\ (8.71) \quad &\leq \mathbb{P}_p\left(\exists G_n \in \mathcal{G}_n \text{ such that } \mathfrak{d}(\mu_n, \nu_{F_j}) \leq \epsilon_{F_j} \text{ and } |\partial^\omega G_n| \leq (1 + \delta')^2 (1 + \xi) n^{d-1} \mathcal{I}_{p,d}(F_j)(\varphi_{F_j^\xi})^{-1} \varphi_{W_{p,d}^\xi}\right) \end{aligned}$$

$$(8.72) \quad \leq \mathbb{P}_p\left(\exists G_n \in \mathcal{G}_n \text{ such that } \mathfrak{d}(\mu_n, \nu_{F_j}) \leq \epsilon_{F_j} \text{ and } |\partial^\omega G_n| \leq (1 + \delta')^2 (1 + \xi) n^{d-1} \mathcal{I}_{p,d}(F_j) \frac{\mathcal{I}_{p,d}(W_{p,d})}{\mathcal{I}_{p,d}(F_j^\xi)}\right)$$

We now apply Theorem 8.9, or more specifically, the consequence (8.42).

(8.73)

$$\mathbb{P}_p(\mathcal{F}_j) \leq \mathbb{P}_p\left(\exists G_n \in \mathcal{G}_n \text{ such that } \mathfrak{d}(\mu_n, \nu_{F_j}) \leq \epsilon_{F_j} \text{ and } |\partial^\omega G_n| \leq (1 + \delta')^2 \frac{1 + \xi}{1 + c(p, d)\epsilon^2} n^{d-1} \mathcal{I}_{p,d}(F_j)\right)$$

And it is from this formula that we determine the relationship (8.45) between ξ and ϵ given ζ . We take δ' smaller if necessary so that (8.69) holds in Case II also. We return to (8.60) and apply this bound (8.69) to each summand.

$$(8.74) \quad \mathbb{P}_p(\exists G_n \in \mathcal{G}_n \text{ such that } \mathfrak{d}(\mu_n, \mathcal{W}_{p,d,\xi}) \geq \epsilon)$$

$$(8.75) \quad \leq \sum_{i=1}^m \mathbb{P}_p\left(\exists G_n \in \mathcal{G}_n \text{ such that } \mathfrak{d}(\mu_n, \nu_{F_j}) \leq \epsilon_{F_j} \text{ and } |\partial^\omega G_n| \leq \lambda_j n^{d-1}\right) + c_1 \exp(-c_2 \log^2 n)$$

$$(8.76) \quad \leq m c_1 \exp(-c_2 n^{1/2d}) + c_1 \exp(-c_2 n^{(1-\epsilon(d))/2d})$$

Where we have used the bounds in Proposition 8.8, and taken the maximum over the finitely many constants c_1 as well as the minimum over the finitely many constants c_2 . An application of Borel-Cantelli yields that

$$(8.77) \quad \mathbb{P}_p \left(\max_{G_n \in \mathcal{G}_n} \mathfrak{d}(\mu_n, \mathcal{W}_{p,d,\xi}) \leq \epsilon(\zeta, \xi) \text{ for all but finitely many } n \right) = 1$$

Notice that

$$(8.78) \quad \mathfrak{d}(\mathcal{W}_{p,d,\xi}, \mathcal{W}_{p,d}) \leq c(p, d) \max \left(\mathcal{L}^d(W_{p,d} \setminus W_{p,d}^\eta), \mathcal{L}^d(W_{p,d}^\xi \setminus W_{p,d}) \right)$$

$$(8.79) \quad \leq c(p, d, \xi)$$

with $c(p, d, \xi) \rightarrow 0$ as $\xi \rightarrow 0$. From (8.45), we have that $\xi = \xi(\zeta) \rightarrow 0$ as $\zeta \rightarrow 0$ and that $\epsilon = \epsilon(\zeta, \xi) \rightarrow 0$ as $\zeta, \xi \rightarrow 0$. Thus,

$$(8.80) \quad \mathbb{P}_p \left(\max_{G_n \in \mathcal{G}_n} \mathfrak{d}(\mu_n, \mathcal{W}_{p,d}) \leq c(p, d, \zeta) \text{ for all but finitely many } n \right) = 1$$

where $c(p, d, \zeta) \rightarrow 0$ as $\zeta \rightarrow 0$. This completes the proof of Theorem 2.2. \square

Proof of Theorem 1.3. Let us first show that $\mathcal{W}_{p,d}$ is compact (we need only show $\mathcal{W}_{p,d}$ is sequentially closed). Let ν_{F_n} be a sequence in $\mathcal{W}_{p,d}$ which is convergent with respect to the metric \mathfrak{d} , so that each F_n is a translate of $W_{p,d}$ contained in $[-1, 1]^d$. The sequence of indicator functions $\mathbf{1}_{F_n}$ converges in L^1 -sense to some $\mathbf{1}_F$, and the corresponding ν_F is necessarily the \mathfrak{d} -limit of the ν_{F_n} . We conclude $\mathcal{L}^d(F) = \mathcal{L}^d(W_{p,d})$ from dominated convergence. Lower semicontinuity (Lemma A.13) of the surface energy tells us $\mathcal{I}_{p,d}(F) \leq \mathcal{I}_{p,d}(W_{p,d})$, which implies F must itself be a translate of $W_{p,d}$ by Theorem 2.3.

Let $\zeta' > 0$, and for each $\nu_W \in \mathcal{W}_{p,d}$, choose ϵ_W as in Proposition 8.8 for W and the parameter $\lambda_W = (1 - \zeta')\mathcal{I}_{p,d}(W_{p,d})$. The \mathfrak{d} -balls $\mathcal{B}(\nu_W, \epsilon_W/2)$ indexed by $\mathcal{W}_{p,d}$ which form an open cover of $\mathcal{W}_{p,d}$. Extract from this open cover a finite collection W_1, \dots, W_m of translates of $W_{p,d}$ such that $\{\mathcal{B}(\nu_{W_i}, \epsilon_{W_i}/2)\}_{i=1}^m$ covers $\mathcal{W}_{p,d}$. Choose $\zeta > 0$ small enough so that the term $c(p, d, \zeta)$ from (8.80) is less than or equal to $\min_{i=1}^m \epsilon_{W_i}/2$, and work within the almost sure event from (8.80) that

$$(8.81) \quad \left\{ \max_{G_n \in \mathcal{G}_n} \mathfrak{d}(\mu_n, \mathcal{W}_{p,d}) \leq c(p, d, \zeta) \text{ for all but finitely many } n \right\}$$

By combining Borel-Cantelli with Proposition 8.8, we may also conclude that each of the following events occurs with full probability:

$$(8.82) \quad \left\{ \text{For only finitely many } n, \exists G_n \in \mathcal{G}_n \text{ such that } |\partial^\omega G_n| \leq (1 - \zeta')\mathcal{I}_{p,d}(W_{p,d}) \text{ and } \mathfrak{d}(\mu_n, \nu_{W_i}) \leq \epsilon_{W_i} \right\}$$

Within the intersection of these almost sure events, it follows that for all n is sufficiently large (depending on ω), we have that each μ_n is such that $\mathfrak{d}(\mu_n, \nu_{W_i}) \leq \epsilon_{W_i}$ for some i , and hence that $|\partial^\omega G_n| \geq (1 - \zeta')\mathcal{I}_{p,d}(W_{p,d})$ for all $G_n \in \mathcal{G}_n$. We conclude that

$$(8.83) \quad \mathbb{P}_p \left(\liminf_{n \rightarrow \infty} \min_{G_n \in \mathcal{G}_n} \frac{|\partial^\omega G_n|}{n^{d-1}} \geq (1 - \zeta')\mathcal{I}_{p,d}(W_{p,d}) \right) = 1$$

Moreover, we lose no generality taking ζ smaller if necessary, in a way depending on p, d and ζ' , so that from (8.81) we have also that

$$(8.84) \quad \mathbb{P}_p \left(\limsup_{n \rightarrow \infty} \max_{G_n \in \mathcal{G}_n} \frac{|G_n|}{n^d} \leq \mathcal{L}^d(W_{p,d}) + \zeta' \right) = 1$$

Let $\epsilon > 0$. We may choose ζ' sufficiently small depending on ϵ so that on the intersection of the events (8.83) and (8.84), we have

$$(8.85) \quad \mathbb{P}_p \left(\liminf_{n \rightarrow \infty} n\widehat{\Phi}_n \geq (1 - \epsilon) \frac{\mathcal{I}_{p,d}(W_{p,d})}{\theta_p(d)\mathcal{L}^d(W_{p,d})} \right) = 1$$

The complimentary upper bound on $\widehat{\Phi}_n$ was shown in Corollary 5.6, so the proof is complete. \square

Proof of Theorem 1.2. Recall that at the end of Section 5, we defined the empirical measure of a translate $W \subset [-1, 1]^d$ of the Wulff crystal as follows:

$$(8.86) \quad \bar{\nu}_W(n) := \frac{1}{n^d} \sum_{x \in \mathbf{C}_n \cap nW} \delta_{x/n}$$

Let $M_n = n^{-1}\mathbb{Z}^d \cap [-1, 1]^d$ and let $\epsilon > 0$. As $|M_n| \leq (3n)^d$, it follows from Corollary 5.7 that

$$(8.87) \quad \mathbb{P}_p \left(\max_{x \in M_n, W_{p,d}+x \subset [-1, 1]^d} \mathfrak{d}(\bar{\nu}_{W_{p,d}+x}, \nu_{W_{p,d}+x}) \leq \epsilon \right) \geq 1 - c_1 \exp(-c_2 n^{d-1})$$

where $c_1(p, d, \epsilon)$ and $c_2(p, d, \epsilon)$ are positive constants. Let us work within this high probability event. We may take n sufficiently large depending on ϵ so that for any translate $W \subset [-1, 1]^d$ of the Wulff crystal, there is $x \in M_n$ so that $\mathfrak{d}(\nu_W, \nu_{W_{p,d}+x}) \leq \epsilon$. Let us also work within the high probability event complimentary to (8.76), so that together with the observation (8.79), and our parameters chosen carefully, we have

$$(8.88) \quad \max_{G_n \in \mathcal{G}_n} \mathfrak{d}(\mu_n, \mathcal{W}_{p,d}) \leq \epsilon$$

within this event. It follows that for all n sufficiently large, on the intersection of the events corresponding to (8.87) and (8.88), we have

$$\max_{G_n \in \mathcal{G}_n} \min_{x \in M_n} \mathfrak{d}(\mu_n, \bar{\nu}_{W_{p,d}+x}(n)) \leq 3\epsilon$$

We use the definition of \mathfrak{d} and apply Borel-Cantelli to complete the proof. \square

APPENDIX A. TOOLS FROM PERCOLATION, GRAPH THEORY AND GEOMETRY

A.1. Tools from percolation. We first present fundamental tools from percolation used both implicitly and explicitly throughout the paper. We use the notation $\Lambda(n) := [-n, n]^d \cap \mathbb{Z}^d$. The first tool was introduced by Benjamini and Mossel in [5], although the proof contained a gap. Mathieu and Remy filled this gap in [45], and alternate proofs were also given by Berger, Biskup, Hoffman and Kozma [7] and by Pete [49]. The version we present is Proposition A.2 of [7].

Proposition A.1. *Let $d \geq 2$ and $p > p_c(d)$. There exist positive constants $c_1(p, d)$, $c_2(p, d)$ and $c_3(p, d)$ so that for all $t > 0$,*

$$\mathbb{P}_p(\exists \Lambda \ni 0, \omega\text{-connected}, |\Lambda| \geq t^{d/(d-1)}, |\partial^\omega \Lambda| < c_3 |\Lambda|^{(d-1)/d}) \leq c_1 \exp(-c_2 t)$$

We deduce a corollary in the same vein as Proposition A.1 in [7].

Corollary A.2. *Let $d \geq 2$ and $p > p_c(d)$. There are positive constants $c_1(p, d)$, $c_2(p, d)$, $c_3(p, d)$ and an almost surely finite random variable $R = R(\omega)$ such that whenever $n \geq R$, we the following lower bound on $\partial^\omega \Lambda$ for each ω -connected Λ satisfying $\Lambda \subset \mathbf{C}_n$ and $|\Lambda| \geq n^{1/2}$:*

$$|\partial^\omega \Lambda| \geq c_3 |\Lambda|^{(d-1)/d}$$

Moreover, we have the following tail bounds on R :

$$\mathbb{P}_p(R > n) \leq c_1 n^d \exp(-c_2 n^{(d-1)/2d})$$

Proof. Let c_3 be as in Proposition A.1, and let \mathcal{E}_n denote the following event:

$$(A.1) \quad \{\exists \Lambda, \omega\text{-connected with } \Lambda \subset \mathbf{C}_n, |\Lambda| \geq n^{1/2} \text{ but } |\partial^\omega \Lambda| < c_3 |\Lambda|^{(d-1)/d}\}$$

We apply Proposition A.1 to every point within the box $\Lambda(n)$ with $t = n^{1/2}$ to obtain

$$(A.2) \quad \mathbb{P}_p(\mathcal{E}_n) \leq c_1 n^d \exp(-c_2 n^{(d-1)/2d})$$

These probabilities are summable in n . We let R be the (random) smallest natural number such that that $n \geq R$ implies \mathcal{E}_n^c occurs. As $\{R > n\} \subset \mathcal{E}_n$, the proof is complete. \square

Proposition A.1 and its corollary give us control on the open edge boundary of large subgraphs of \mathbf{C}_n . We now introduce a tool for controlling the density of the infinite cluster within a large box. This result was proved in two dimensions by Durrett and Schonmann [30] and in higher dimensions in the thesis of Gandolfi [32].

Proposition A.3. *Let $d \geq 2$ and $p > p_c(d)$. Recall that $\theta_p(d) = \mathbb{P}_p(0 \in \mathbf{C}_\infty)$ is the density of the infinite cluster. For any $\epsilon > 0$, there exist positive constants $c_1(p, d, \epsilon)$ and $c_2(p, d, \epsilon)$ so that*

$$\mathbb{P}_p\left(\frac{|\mathbf{C}_n|}{|\Lambda(n)|} \notin (\theta_p(d) - \epsilon, \theta_p(d) + \epsilon)\right) \leq c_1 \exp(-c_2 n^{d-1})$$

It is interesting to note that the upper deviations actually decay exponentially with n^d , whereas the lower deviations are of surface order, hence the surface term in the above proposition. Pisztora [52] later refined these results, showing that with high probability \mathbf{C}_n consists of a unique giant connected component whose volume is roughly $\theta_p |\Lambda(n)|$ with all other connected components of negligible size. Pisztora worked in the more general setting of the FK percolation and the Ising model, and even showed this giant component has important geometric properties, spanning each of the opposing faces of $\Lambda(n)$. These properties are useful for renormalization arguments, but we will not need to be so delicate for our applications. The following is an immediate corollary of Proposition A.3.

Corollary A.4. *Let $d \geq 2$ and $p > p_c(d)$. Let $r > 0$, let $Q \subset \mathbb{R}^d$ be a translate of the cube $[-r, r]^d$ and let $\epsilon > 0$. There exist positive constants $c_1(p, d, \epsilon)$, $c_2(p, d, \epsilon)$ so that*

$$\mathbb{P}\left(\frac{|\mathbf{C}_\infty \cap Q|}{\mathcal{L}^d(Q)} \notin (\theta_p(d) - \epsilon, \theta_p(d) + \epsilon)\right) \leq c_1 \exp(-c_2 r^{d-1})$$

Finally, in Section 8, we must use an important result of Kesten, Zhang [43], Grimmett and Marstrand [36].

Theorem A.5. *Let $d \geq 2$ and $p > p_c(d)$, and let $\mathbf{C}(0)$ denote the open cluster containing the origin. There is a positive constant $c(p)$ so that*

$$\mathbb{P}_p(|\mathbf{C}(0)| = n) \leq \exp(-cn^{(d-1)/d})$$

A.2. Using tools from percolation. We now specialize the tools of Appendix A.1 to the setting of our problem. Recall that \mathcal{G}_n is the collection of Cheeger optimizers for $\widehat{\Phi}_n$. We begin by making a basic observation.

Lemma A.6. *For all n and each configuration ω , if $G_n \in \mathcal{G}_n$ and is disconnected, then G_n is a finite union of connected optimal subgraphs.*

Proof. The proof follows from the identity that for $a, b, c, d > 0$, we have

$$(A.3) \quad \frac{a+b}{c+d} \geq \min\left(\frac{a}{c}, \frac{b}{d}\right)$$

By the discussion at the end of Section 2.2, it must be that the connected components of any $G_n \in \mathcal{G}_n$ have disjoint open edge boundaries. Thus if G_n is optimal and disconnected, and if we decompose G into two disjoint subgraphs G'_n and G''_n , we must have $\varphi_{G_n} = \varphi_{G'_n} = \varphi_{G''_n}$. \square

We now use Corollary A.4 to obtain a high probability upper bound on $\widehat{\Phi}_n$.

Lemma A.7. *Let $d \geq 2$ and $p > p_c(d)$. There are positive constants $c_1(p, d)$, $c_2(p, d)$ and $c'_3(p, d)$ so that*

$$\mathbb{P}_p(\widehat{\Phi}_n > c'_3 n^{-1}) \leq c_1 \exp(-c_2 n^{d-1})$$

Proof. We abbreviate $\theta_p(d)$ as θ . Let us work within the high probability event from Corollary A.4 for the box $[-r, r]^d$, where $r = n/2(d!)^{1/d}$, and for the parameter $\epsilon > 0$. Let us also work within the corresponding high probability event for the box $[-n, n]^d$ with the same parameter ϵ . Let $\epsilon' = \epsilon/\theta$ and let H_n be $\mathbf{C}_\infty \cap [-r, r]^d$, so that

$$(A.4) \quad (1 - \epsilon')\theta(2r)^d \leq |C| \leq (1 + \epsilon')\theta(2r)^d$$

$$(A.5) \quad \leq (1 - \epsilon')\theta(2n)^d$$

For ϵ chosen sufficiently small depending on d . Thus, H_n is valid with volume on the order of n^d . The open edge boundary of H_n is at most a constant depending on d times the \mathcal{H}^{d-1} -measure of $\partial[-r, r]^d$, which completes the proof. \square

The above result may be used in conjunction with Corollary A.2 to obtain a high probability lower bound on the volume of any Cheeger optimizer.

Lemma A.8. *Let $d \geq 2$ and $p > p_c(d)$. There exist positive constants $c_1(p, d)$, $c_2(p, d)$ and $\eta_1(p, d)$ so that,*

$$\mathbb{P}_p(\exists G_n \in \mathcal{G}_n \text{ such that } |G_n| < \eta_1 n^d) \leq c_1 \exp(-c_2 n^{(d-1)/2d})$$

Proof. Work within the intersection of the high probability events

$$(A.6) \quad \{\widehat{\Phi}_n \leq c'_3 n^{-1}\} \cap \{R \leq n\}$$

respectively from Lemma A.7 and Corollary A.2. Consider $G_n \in \mathcal{G}_n$, and using Lemma A.6, extract from G_n a connected subgraph $H_n \subset G_n$ with $H_n \in \mathcal{G}_n$. Suppose that $|H_n| \leq n^{1/2}$. That $H_n \subset \mathbf{C}_n$ implies $\partial^\omega H_n$ is non-empty, and hence that $\varphi_{H_n} > n^{-1/2}$. This is impossible when $\widehat{\Phi}_n \leq c'_3 n^{-1}$ and n is sufficiently large.

Thus we may suppose that $H_n \geq n^{1/2}$, and using our event from Corollary A.2, we have

$$(A.7) \quad |\partial^\omega H_n| \geq c_3 |H_n|^{(d-1)/d}$$

Thus,

$$(A.8) \quad c_3 |H_n|^{-1/d} \leq \varphi_{H_n} \leq c'_3 n^{-1}$$

and the claim holds with $\eta_1 = (c_3/c'_3)^d$. \square

Combining Lemma A.8 with Lemma A.6 and the observation that $|\Lambda(n)| \leq c(d)n^d$, we immediately deduce the following.

Corollary A.9. *Let $d \geq 2$ and $p > p_c(d)$. There exist positive constants $c_1(p, d)$, $c_2(p, d)$ and $\eta_4(p, d)$ so that*

$$\mathbb{P}_p \left(\begin{array}{l} \exists G_n \in \mathcal{G}_n \text{ such that the number of} \\ \text{connected components of } G_n \text{ exceeds } \eta_4 \end{array} \right) \leq c_1 \exp(-c_2 n^{(d-1)/2d})$$

Having established that all Cheeger optimizers are, with high probability, volume order subgraph of $\Lambda(n)$, we now exhibit control from above and below on the open edge boundary of each Cheeger optimizer.

Lemma A.10. *Let $d \geq 2$ and $p > p_c(d)$. There exist positive constants $c_1(p, d)$, $c_2(p, d)$ and $\eta_2(p, d)$, $\eta_3(p, d)$ so that*

$$\mathbb{P}_p \left(\exists G_n \in \mathcal{G}_n \text{ so that } |\partial^\omega G_n| < \eta_2 n^{d-1} \text{ or } |\partial^\omega G_n| > \eta_3 n^{d-1} \right) \leq c_1 \exp \left(-c_2 n^{(d-1)/2d} \right)$$

Proof. First work within the high probability event $\{\widehat{\Phi}_n \leq c'_3 n^{-1}\}$ from Lemma A.7 and consider some $G_n \in \mathcal{G}_n$. That $\varphi_{G_n} \leq c'_3 n^{-1}$ and $|G_n| \leq c(d)n^d$ together imply

$$(A.9) \quad |\partial^\omega G_n| \leq c'_3 c(d) n^{d-1}$$

so we set $\eta_3 = c'_3 c(d)$. To prove the second half of this Lemma, we work on the intersection of the high probability events

$$(A.10) \quad \{R \leq n\} \cap \{\forall G_n \in \mathcal{G}_n, \text{ we have } |G_n| \geq \eta_1 n^d\}$$

from Corollary A.2 and Lemma A.8. Given $G_n \in \mathcal{G}_n$, we extract (through Lemma A.6) to $H_n \subset G_n$ with $H_n \in \mathcal{G}_n$ and H_n connected. On the event $\{R \leq n\}$, we have

$$(A.11) \quad |\partial^\omega G_n| \geq |\partial^\omega H_n| \geq c_3 \left(\eta_1 n^d \right)^{(d-1)/d}$$

and we set $\eta_2 = c_3(\eta_1)^{(d-1)/d}$. □

A.3. Tools from graph theory. First, we state the version of Turán's theorem given directly before Lemma 6 in [57]. This theorem is used in the Peierls argument of Section 6. Recall that for a graph (V, E) , an *independent* set of vertices $A \subset V$ is a collection of vertices such that no two elements of A are joined by an edge in E . We suppose that (V, E) is a finite graph. The *independence number* of (V, E) is

$$(A.12) \quad \alpha(V, E) := \max\{|A| : A \text{ is an independent subset of } V\}$$

Theorem A.11. *Let (V, E) be a finite graph with maximal degree δ . Then*

$$\alpha(V, E) \geq \frac{|V|}{\delta + 1}$$

Our next combinatorial proposition gives an exponential bound on the number of \mathbb{L}^d -connected subsets of \mathbb{Z}^d containing the origin.

Proposition A.12. *There is a positive constant $c(d)$ so that the number of \mathbb{L}^d -connected subsets of \mathbb{Z}^d containing the origin of size s is at most $[c(d)]^s$.*

Proof. This is a standard estimate. See the proof of Theorem 4.20, and equation (4.24) in [37]. □

A.4. Approximation and miscellany. This section contains useful information about the surface energy, as well as a proof that a nice chosen rotation (from Section 3) exists. We now state a fundamental property of the surface energy \mathcal{I}_τ associated to any norm τ on \mathbb{R}^d .

Lemma A.13. *Let τ be a norm on \mathbb{R}^d and let \mathcal{I}_τ be the surface energy associated to τ . The surface energy \mathcal{I}_τ is lower semicontinuous, that is, if E_n is a sequence of Borel sets in \mathbb{R}^d tending to E with respect to the metric d_{L^1} , we have*

$$\mathcal{I}_\tau(E) \leq \liminf_{n \rightarrow \infty} \mathcal{I}_\tau(E_n)$$

The proof of Lemma A.13 is immediate from the definition of the surface energy, see Section 14.2 of [18]. We may use this lower semicontinuity to approximate the Wulff crystal (in both volume and surface energy) by polytopes.

Proposition A.14. Consider the Wulff crystal $W_{p,d} \subset [-1, 1]^d$. The Wulff crystal is a set of finite perimeter, and for all $\epsilon > 0$, there is a polytope $P_\epsilon \subset W_{p,d}$ so that

- (i) $|\mathcal{I}_{p,d}(P_\epsilon) - \mathcal{I}_{p,d}(W_{p,d})| \leq \epsilon$
- (ii) $\mathcal{L}^d(W_{p,d} \setminus P_\epsilon) \leq \epsilon$

Proof. The Wulff crystal is a set of finite perimeter because it is convex and bounded (see Proposition 3.6 and Lemma 3.7). For $k \in \mathbb{N}$, consider a collection of points $\{x_1^{(k)}, \dots, x_m^{(k)}\}$ (with m depending on k) in $\partial W_{p,d}$ such that for each $x \in \partial W_{p,d}$, there is some $x_i^{(k)}$ with $|x_i^{(k)} - x|_2 < 2^{-k}$.

For each $k \in \mathbb{N}$, let P_k denote the convex hull of $x_1^{(k)}, \dots, x_m^{(k)}$, and note that when k is sufficiently large, the P_k are non-degenerate polytopes contained in $W_{p,d}$, so that $\text{per}(P_k) \leq \text{per}(W_{p,d})$ for all k . By construction, $\mathcal{L}^d(W_{p,d} \setminus P_k)$ tends to zero as $k \rightarrow \infty$, and we use Lemma A.13 to conclude

$$(A.13) \quad \lim_{k \rightarrow \infty} \text{per}(P_k) \rightarrow \text{per}(W_{p,d})$$

As $\beta_{p,d}$ is a norm on \mathbb{R}^d , there are positive constants $c(\beta_{p,d}, d)$ and $C(\beta_{p,d}, d)$ so that for each $x \in \mathbb{R}^d$, we have $c|x|_2 \leq \beta_{p,d}(x) \leq C|x|_2$. This implies the following inequalities hold for all $k \in \mathbb{N}$:

$$(A.14) \quad \text{cper}(P_k) \leq \mathcal{I}_{p,d}(P_k) \leq C\text{per}(P_k) \quad \text{and} \quad \text{cper}(W_{p,d}) \leq \mathcal{I}_{p,d}(W_{p,d}) \leq C\text{per}(W_{p,d})$$

which completes the proof upon using (A.13). \square

Note that a much more general theorem of this flavor holds for all sets of finite perimeter (see Proposition 14.9 of [18]). We do not use this theorem because it does not guarantee that the polyhedral approximate P_ϵ to a set of finite perimeter F is contained in F , and also because the above proof is so short.

The last subject we deal with in the appendix is the so-called “chosen” rotation from Section 3. Let \mathbb{H}_+^{d-1} denote the closed, upper hemisphere of the unit $(d-1)$ -sphere \mathbb{S}^{d-1} . We denote by $T\mathbb{S}^{d-1}$ the tangent bundle of \mathbb{S}^{d-1} , and the restriction of this bundle to \mathbb{H}_+^{d-1} shall be written as $T\mathbb{H}_+^{d-1}$. An *orthonormal k -frame* on \mathbb{H}_+^{d-1} is an assignment to each point $x \in \mathbb{H}_+^{d-1}$ an ordered collection of k orthonormal vectors in $T_x\mathbb{H}_+^{d-1}$, the tangent plane to \mathbb{H}_+^{d-1} at x . Each such assignment may be expressed in Euclidean coordinates thanks to the natural embedding of $\mathbb{S}^{d-1} \subset \mathbb{R}^d$ and each $T_x\mathbb{S}^{d-1}$ into \mathbb{R}^d , and may thus be written as a function

$$(A.15) \quad f : x \mapsto (v_1(x), \dots, v_k(x))$$

with $x \in \mathbb{H}_+^{d-1} \subset \mathbb{R}^d$, and with each $v_i(x) \in \mathbb{R}^d$. The collection of all such functions f which vary smoothly has the structure of a smooth manifold, denoted $V_k(\mathbb{H}_+^{d-1})$, and called the *Stiefel manifold* for the pair (\mathbb{H}_+^{d-1}, k) . Given $x, y \in \mathbb{R}^d$, let us use $|x - y|$ to denote the Euclidean distance between x and y .

Proposition A.15. There exists an orthonormal $(d-1)$ -frame $f \in V_{d-1}(\mathbb{H}_+^{d-1})$ and a constant $C > 0$ so that for $\epsilon > 0$, whenever $x, y \in \mathbb{H}_+^{d-1}$ satisfy $|x - y| \leq \epsilon$, we have

$$|(f(x))_i - (f(y))_i| \leq C\epsilon$$

for all $i \in \{1, \dots, d-1\}$.

Proof. Let $s \in \mathbb{S}^{d-1}$ denote the south pole, with coordinate representation $(0, \dots, 0, -1)$ in \mathbb{R}^d . Consider the standard stereographic projection map $P : \mathbb{S}^{d-1} \setminus \{s\} \rightarrow \mathbb{R}^{d-1}$, and note that the image of \mathbb{H}_+^{d-1} under this map is a closed disc $D \subset \mathbb{R}^{d-1}$ centered at the origin. This disc D is parallelizable, that is, it is possible to construct a smooth $(d-1)$ -frame g on D . Indeed, one can just take the standard basis for \mathbb{R}^{d-1} at each tangent space $T_y D$. Define $f \in V_{d-1}(\mathbb{H}_+^{d-1})$ as the pullback P^*g . As f varies smoothly over a compact domain, it follows that each of its coordinate functions is Lipschitz, which completes the proof. \square

REFERENCES

- [1] G. Alberti, G. Bellettini, M. Cassandro, and E. Presutti. Surface tension in Ising systems with Kac potentials. *J. Statist. Phys.*, 82(3-4):743–796, 1996.
- [2] K Alexander, JT Chayes, and L Chayes. The wulff construction and asymptotics of the finite cluster distribution for two-dimensional bernoulli percolation. *Communications in mathematical physics*, 131(1):1–50, 1990.
- [3] N. Alon. Eigenvalues and expanders. *Combinatorica*, 6(2):83–96, 1986. Theory of computing (Singer Island, Fla., 1984).
- [4] Peter Antal and Agoston Pisztora. On the chemical distance for supercritical Bernoulli percolation. *Ann. Probab.*, 24(2):1036–1048, 1996.
- [5] Itai Benjamini and Elchanan Mossel. On the mixing time of a simple random walk on the super critical percolation cluster. *Probability Theory and Related Fields*, 125(3):408–420, 2003.
- [6] O Benois, T Bodineau, P Butta, and E Presutti. On the validity of van der waals theory of surface tension. *Markov Process. Related Fields*, 3(2):175–198, 1997.
- [7] Noam Berger, Marek Biskup, Christopher E Hoffman, and Gady Kozma. Anomalous heat-kernel decay for random walk among bounded random conductances. In *Annales de l’IHP Probabilités et statistiques*, volume 44, pages 374–392, 2008.
- [8] Marek Biskup, Oren Louidor, Eviatar B. Procaccia, and Ron Rosenthal. Isoperimetry in two-dimensional percolation. *Communications on Pure and Applied Mathematics*, 68(9):1483–1531, 2015.
- [9] Viktor Blåsjö. The isoperimetric problem. *The American Mathematical Monthly*, 112(6):526–566, 2005.
- [10] T Bodineau. The wulff construction in three and more dimensions. *Communications in mathematical physics*, 207(1):197–229, 1999.
- [11] T. Bodineau. On the van der Waals theory of surface tension. *Markov Process. Related Fields*, 8(2):319–338, 2002. Inhomogeneous random systems (Cergy-Pontoise, 2001).
- [12] T. Bodineau, D. Ioffe, and Y. Velenik. Winterbottom construction for finite range ferromagnetic models: an L_1 -approach. *J. Statist. Phys.*, 105(1-2):93–131, 2001.
- [13] Thierry Bodineau, Dmitri Ioffe, and Yvan Velenik. Rigorous probabilistic analysis of equilibrium crystal shapes. *Journal of Mathematical Physics*, 41(3):1033–1098, 2000.
- [14] Andrea Braides and Andrey Piatnitski. Variational problems with percolation: dilute spin systems at zero temperature. *Journal of Statistical Physics*, 149(5):846–864, 2012.
- [15] Andrea Braides and Andrey Piatnitski. Homogenization of surface and length energies for spin systems. *Journal of Functional Analysis*, 264(6):1296–1328, 2013.
- [16] R. M. Burton and M. Keane. Density and uniqueness in percolation. *Comm. Math. Phys.*, 121(3):501–505, 1989.
- [17] Raphaël Cerf. Large deviations for three dimensional supercritical percolation. *Astérisque*, (267):vi+177, 2000.
- [18] Raphaël Cerf. *The Wulff Crystal in Ising and Percolation Models: Ecole d’Eté de Probabilités de Saint-Flour XXXIV-2004*, volume 34. Springer, 2006.
- [19] Raphaël Cerf and Ágoston Pisztora. On the wulff crystal in the ising model. *Annals of probability*, pages 947–1017, 2000.
- [20] Raphaël Cerf and Ágoston Pisztora. Phase coexistence in ising, potts and percolation models. In *Annales de l’IHP Probabilités et statistiques*, volume 37, pages 643–724, 2001.
- [21] Raphaël Cerf and Marie Thérét. Law of large numbers for the maximal flow through a domain of \mathbb{R}^d in first passage percolation. *Transactions of the American Mathematical Society*, 363(7):3665–3702, 2011.
- [22] Raphaël Cerf and Marie Thérét. Lower large deviations for the maximal flow through a domain of \mathbb{R}^d in first passage percolation. *Probab. Theory Related Fields*, 150(3-4):635–661, 2011.
- [23] Raphaël Cerf and Marie Thérét. Maximal stream and minimal cutset for first passage percolation through a domain of \mathbb{R}^d . *Annals of Probability 2014, Vol. 42, No. 3, 1054-1120*, 01 2012.
- [24] Jeff Cheeger. A lower bound for the smallest eigenvalue of the Laplacian. In *Problems in analysis (Papers dedicated to Salomon Bochner, 1969)*, pages 195–199. Princeton Univ. Press, Princeton, N. J., 1970.
- [25] Fan R. K. Chung. *Spectral graph theory*, volume 92 of *CBMS Regional Conference Series in Mathematics*. Published for the Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI, 1997.
- [26] J. Theodore Cox and Richard Durrett. Some limit theorems for percolation processes with necessary and sufficient conditions. *Ann. Probab.*, 9(4):583–603, 1981.
- [27] Jean-Dominique Deuschel and Ágoston Pisztora. Surface order large deviations for high-density percolation. *Probab. Theory Related Fields*, 104(4):467–482, 1996.
- [28] RL Dobrushin, R Kotecký, and SB Shlosman. A microscopic justification of the wulff construction. *Journal of statistical physics*, 72(1-2):1–14, 1993.
- [29] Roland Lvoř Dobrushin, Roman Kotecký, and S Shlosman. *Wulff construction: a global shape from local interaction*, volume 104. American Mathematical Society Providence, Rhode Island, 1992.
- [30] R. Durrett and R. H. Schonmann. Large deviations for the contact process and two-dimensional percolation. *Probab. Theory Related Fields*, 77(4):583–603, 1988.

- [31] Alessio Figalli, Francesco Maggi, and Aldo Pratelli. A mass transportation approach to quantitative isoperimetric inequalities. *Inventiones mathematicae*, 182(1):167–211, 2010.
- [32] Alberto Gandolfi. *Clustering and uniqueness in mathematical models of percolation phenomena*. PhD thesis, Technische Universiteit Delft, 1989.
- [33] Olivier Garet. Capacitive flows on a 2d random net. *The Annals of Applied Probability*, pages 641–660, 2009.
- [34] Olivier Garet, Régine Marchand, Eviatar B Procaccia, and Marie Thérét. Continuity of the time and isoperimetric constants in supercritical percolation. *arXiv preprint arXiv:1512.00742*, 2015.
- [35] Josiah Willard Gibbs. On the equilibrium of heterogeneous substances. *American Journal of Science*, (96):441–458, 1878.
- [36] G. R. Grimmett and J. M. Marstrand. The supercritical phase of percolation is well behaved. *Proc. Roy. Soc. London Ser. A*, 430(1879):439–457, 1990.
- [37] Geoffrey Grimmett. *Percolation*, volume 321 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, second edition, 1999.
- [38] Geoffrey Grimmett and Harry Kesten. First-passage percolation, network flows and electrical resistances. *Zeitschrift für Wahrscheinlichkeitstheorie und verwandte Gebiete*, 66(3):335–366, 1984.
- [39] Dmitry Ioffe. Exact large deviation bounds up to T_c for the ising model in two dimensions. *Probability theory and related fields*, 102(3):313–330, 1995.
- [40] Dmitry Ioffe and Roberto H Schonmann. Dobrushin–Kotecký–Shlosman theorem up to the critical temperature. *Communications in mathematical physics*, 199(1):117–167, 1998.
- [41] Harry Kesten. Aspects of first passage percolation. In *École d’été de probabilités de Saint-Flour, XIV—1984*, volume 1180 of *Lecture Notes in Math.*, pages 125–264. Springer, Berlin, 1986.
- [42] Harry Kesten. Surfaces with minimal random weights and maximal flows: a higher-dimensional version of first-passage percolation. *Illinois J. Math.*, 31(1):99–166, 1987.
- [43] Harry Kesten and Yu Zhang. The probability of a large finite cluster in supercritical Bernoulli percolation. *Ann. Probab.*, 18(2):537–555, 1990.
- [44] T. M. Liggett, R. H. Schonmann, and A. M. Stacey. Domination by product measures. *Ann. Probab.*, 25(1):71–95, 1997.
- [45] Pierre Mathieu and Elisabeth Remy. Isoperimetry and heat kernel decay on percolation clusters. *Ann. Probab.*, 32(1A):100–128, 2004.
- [46] R. A. Minlos and Ja. G. Sinai. The phenomenon of “separation of phases” at low temperatures in certain lattice models of a gas. I. *Mat. Sb. (N.S.)*, 73 (115):375–448, 1967.
- [47] R. A. Minlos and Ja. G. Sinai. The phenomenon of “separation of phases” at low temperatures in certain lattice models of a gas. II. *Trudy Moskov. Mat. Obšč.*, 19:113–178, 1968.
- [48] Ben Morris and Yuval Peres. Evolving sets, mixing and heat kernel bounds. *Probability Theory and Related Fields*, 133(2):245–266, 2005.
- [49] Gábor Pete. A note on percolation on \mathbb{Z}^d : isoperimetric profile via exponential cluster repulsion. *Electron. Commun. Probab.*, 13:377–392, 2008.
- [50] C-E Pfister and Yvan Velenik. Large deviations and continuum limit in the 2d ising model. *Probability theory and related fields*, 109(4):435–506, 1997.
- [51] C-E Pfister and Yvan Alain Velenik. Mathematical theory of the wetting phenomenon in the 2d ising model. *Helvetica Physica Acta*, 69:949–973, 1996.
- [52] Agoston Pisztora. Surface order large deviations for Ising, Potts and percolation models. *Probab. Theory Related Fields*, 104(4):427–466, 1996.
- [53] Eviatar B. Procaccia and Ron Rosenthal. Concentration estimates for the isoperimetric constant of the supercritical percolation cluster. *Electron. Commun. Probab.*, 17:no. 30, 11, 2012.
- [54] Clément Rau. Sur le nombre de points visités par une marche aléatoire sur un amas infini de percolation. *Bull. Soc. Math. France*, 135(1):135–169, 2007.
- [55] Raphaël Rossignol and Marie Thérét. Law of large numbers for the maximal flow through tilted cylinders in two-dimensional first passage percolation. *Stochastic Processes and their Applications*, 120(6):873–900, 2010.
- [56] Raphaël Rossignol and Marie Thérét. Lower large deviations and laws of large numbers for maximal flows through a box in first passage percolation. *Ann. Inst. Henri Poincaré Probab. Stat.*, 46(4):1093–1131, 2010.
- [57] J. Michael Steele and Yu Zhang. Nondifferentiability of the time constants of first-passage percolation. *Ann. Probab.*, 31(2):1028–1051, 2003.
- [58] Michel Talagrand. Concentration of measure and isoperimetric inequalities in product spaces. *Publications Mathématiques de l’Institut des Hautes Etudes Scientifiques*, 81(1):73–205, 1995.
- [59] Jean E. Taylor. Existence and structure of solutions to a class of nonelliptic variational problems. In *Symposia Mathematica, Vol. XIV (Convegno di Teoria Geometrica dell’Integrazione e Varietà Minimali, INDAM, Roma, Maggio 1973)*, pages 499–508. Academic Press, London, 1974.

- [60] Jean E. Taylor. Unique structure of solutions to a class of nonelliptic variational problems. In *Differential geometry (Proc. Sympos. Pure. Math., Vol. XXVII, Stanford Univ., Stanford, Calif., 1973), Part 1*, pages 419–427. Amer. Math. Soc., Providence, R.I., 1975.
- [61] Jean E. Taylor. Crystalline variational problems. *Bull. Amer. Math. Soc.*, 84(4):568–588, 1978.
- [62] Marie Theret. Upper large deviations for maximal flows through a tilted cylinder. *ESAIM: Probability and Statistics*, 18:117–129, 2014.
- [63] Ádám Timár. Boundary-connectivity via graph theory. *Proc. Amer. Math. Soc.*, 141(2):475–480, 2013.
- [64] WL Winterbottom. Equilibrium shape of a small particle in contact with a foreign substrate. *Acta Metallurgica*, 15(2):303–310, 1967.
- [65] G Wulff. Zur frage der geschwindigkeit des wachstums und der auflosung der kristallflächen. *Z. Kryst. Miner*, 34:449, 1901.
- [66] Christopher Zach, Marc Niethammer, and Jan-Michael Frahm. Continuous maximal flows and wulff shapes: Application to mrfs. In *Computer Vision and Pattern Recognition, 2009. CVPR 2009. IEEE Conference on*, pages 1911–1918. IEEE, 2009.
- [67] Yu Zhang. Limit theorems for maximum flows on a lattice. 10 2007.
- [68] RKP Zia. Anisotropic surface tension and equilibrium crystal shapes. *Progress in statistical mechanics. World Scientific, Singapore*, pages 303–357, 1988.